Some problems related to grid n-ogons *

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Abstract

In this paper we study some problems related to *grid n-ogons*. A grid n-ogon is a n-vertex orthogonal simple polygon, with no collinear edges, that may be placed in a $(\frac{n}{2}) \times (\frac{n}{2})$ square grid. We will present some problems and results related to a subclass of grid n-ogons, the THIN grid n-ogons, in particular a classification for theses subclass of polygons. We follow presenting the solution of the MINIMUM VERTEX GUARD problem for the MIN-AREA and for the SPIRAL grid n-ogons. Finally the solution of the MAXIMUM HIDDEN VERTEX SET problem for THIN grid n-ogons is also presented.

1 Introduction

In the field of visibility problems, *guarding* and *hiding* are among the most distinguished and exhaustively studied problems. In visibility problems we are given as input a *simple polygon* (simple closed polygonal curve with its interior). In guarding we need to find a minimum number of guards positions in the polygon, such that these guards collectively see the whole polygon. Two points in the polygon see each other, if the line segment connecting them does lies entirely in the polygon. In hiding, we need to find a maximum number of positions in the polygon, such that no two of these positions see each other.

The guarding problems started during a conference, in 1976, when Victor Klee, posed the following problem, which today is known as the *original art gallery problem*: How many stationary guards are needed to guard an art gallery room with n walls? In the abstract version of this problem, the input is a simple polygon P in the plane, representing the floor plan of the art gallery room and a guard is considered a fixed point in P with 2π range visibility. A set of guards covers P, if each point of P is seen by at least one guard. Many variations of the original art gallery theorem have been studied over the years, such as: where the guards may be positioned (anywhere or in specific positions, e.g., vertices) what kind of guards are to be used (e.g., stationary guards versus mobile guards) and what assumptions are on the input polygon (such as being orthogonal) (see [9]). The "opposite" problem of hiding a maximum number of objects from each other in a given simple polygon can have a practical application in computer-games, where a player needs to find and collect or destroy as many objects as possible. Not seeing the next object while collecting an object makes the game more interesting. Such as the guarding problems, this problem has many variations [2].

In this paper, of the guarding problems, we will consider the MINIMUM VERTEX GUARD (MVG) problem, that is the problem of finding the minimum number of guards placed on vertices (*vertex guards*) needed to guard a given simple polygon. And of the hiding problems, we will consider the

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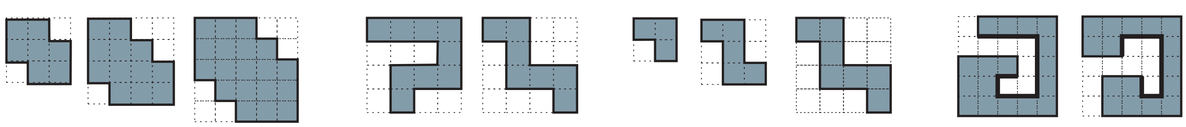
Maximum Hidden Set Vertex (MHVS) problem, that is the problem of finding the maximum number of vertices of a given simple, such that no two vertices see each other. Both problems are NP-hard [3, 7]. An important subclass of polygons are the *orthogonal simple polygons* (simple polygons whose edges meet at right angles). Indeed, they are useful as approximations to polygons; and they arise naturally in domains dominated by Cartesian coordinates, such as raster graphics, VLSI design, or architecture. The MVG and MHVS problems are still NP-hard for orthogonal polygons.

Our contribution: This paper has as intention to introduce a particular type of orthogonal polygons - *the grid n-ogons* - that presents sufficiently interesting characteristics that we are studying and formalizing. Of the problems related to grid n-ogons, the visibility problems are the ones that motivate us more, particulary guarding and hiding problems. The paper is structured as follows: in the next subsection we will introduce some introductory definitions and useful results. In section 3 we will present some problems and results related to Thin grid n-ogons, in particular a classification for these polygons. In section 4 we will expose some results related to the MVG problem on grid n-ogons and we will study the MHVS problem on Thin grid n-ogons, a subclass of grid n-ogons. Finally, in section 5 we will draw conclusions.

2 Conventions, Definitions and Results

In this paper, the interior and the boundary of a simple polygon P will be denoted by $\textsc{Int}(P)$ and $\textsc{Bnd}(P)$, respectively. And, for convenience, we will assume that the vertices of P are ordered in a counterclockwise (CCW) direction around $\textsc{Int}(P)$. A vertex of P is called convex if the interior angle between its two incident edges is at most π, otherwise is called reflex. We use r to represent the number of reflex vertices of P. It has been shown by O'Rourke that $n = 2r + 4$, for every orthogonal simple polygon of n vertices (n-ogon, for short). A *rectilinear cut* of a n-ogon P is a partition of P obtained by extending each edge incident to a reflex vertex of P towards $\textsc{Int}(P)$ until it hits $\textsc{Bnd}(P)$. We denote this partition by $\Pi(P)$ and the number of its pieces by $|\Pi(P)|$. Each piece is a rectangle and so we call it a *r-piece*. A n-ogon that may be placed in a $\frac{n}{2} \times \frac{n}{2}$ square grid and that does not have collinear edges is called *grid n-ogon*. We assume that the grid is defined by the horizontal lines $y = 1, \ldots, y = \frac{n}{2}$ and the vertical lines $x = 1, \ldots, x = \frac{n}{2}$ and that its northwest corner is $(1, 1)$. Each grid n-ogon has exactly one edge in every line of the grid. Grid n-ogons that are symmetrically equivalent are grouped in the same class [1]. A grid n-ogon Q is called Fat iff $|\Pi(Q)| \geq |\Pi(P)|$, for all grid n-ogons P. Similarly, a grid n-ogon Q is called Thin iff $|\Pi(Q)| \leq |\Pi(P)|$, for all grid n-ogons P. Let P be a grid n-ogon and r the number of its reflex vertices. In [1] is proved that, if P is Fat then $|\Pi(P)| = \frac{3r^2+6r+4}{4}$, for r even and $|\Pi(P)| = \frac{3(r+1)^2}{4}$, for r odd; if P is Thin then $|\Pi(P)| = 2r + 1$. There is a single Fat grid n-ogon (see Fig. 1 (a)); however, Thin grid n-ogons are not unique (see Fig. 1 (b)).



(a) (b) (c) (d)

Figure 1: (a) The unique Fat grid n-ogons, for $r = 2, 3, 4$; (b) Two Thin 10-ogons; (c) The unique Min-Area grid n-ogons, for $r = 1, 2, 3$; (d) Two different Spiral grid 12-ogons (reflex chain in bold).

The area of a grid n-ogon is the number of grid cells in its interior. In [1] it is proved that for all grid n-ogon P, with $n \geq 8$, $2r + 1 \leq A(P) \leq r^2 + 3$. A grid n-ogon P is a Max-Area grid n-ogon iff $A(P) = r^2 + 3$ and it is a Min-Area grid n-ogon iff $A(P) = 2r + 1$. There are Max-Area grid n-ogons for all n, but they are not unique. However, there is a single Min-Area grid n-ogon and it has the form illustrated in Fig. 1 (c). Regarding Min-Area grid n-ogons, it is obvious that they are Thin grid n-ogons, because $|\Pi(P)| = 2r+1$ holds only for Thin grid n-ogons. However, this condition is not sufficient for a grid n-ogon to be a Min-Area grid n-ogon. A grid n-ogon is called Spiral grid n-ogon if its boundary can be divided into a reflex chain and a convex chain. A polygonal chain is called *reflex* if its vertices are all reflex (all except the vertices at the end of the chain) with respect to the interior of the polygon. And, a polygonal chain is called *convex* if its vertices are all convex with respect to the interior of the polygon. In [6] is proved that that there are SPIRAL grid n-ogons, for all $n \geq 6$; however, they are not unique, as we may see in Fig.1 (d). And it is also proved that every SPIRAL grid n-ogons, with $r \geq 1$ reflex vertices, is a THIN grid n-ogon.

Given a n-ogon P, we can associate to $\Pi(P)$ a graph, denominated by *dual graph* of $\Pi(P)$ and denoted by $G(P)$, which captures the adjacency relation between pieces of the partition. Each node of the dual graph corresponds to a piece of the partition and its non-oriented edges connect adjacent pieces, i.e., pieces with a common edge. We prove that if P is a THIN grid n-ogon then $G(P)$ is a path graph, i.e., a tree with two nodes of vertex degree 1, called leaves, and the other nodes of vertex degree 2. To prove this result we introduce lemma 2.1.

Lemma 2.1. *Let P be a* THIN *$(n+2)$-ogon. Then every grid n-ogon that yields P by* INFLATE-PASTE *(a correct and complete method to generate grid n-ogons, well described in [8]) is also a* THIN.

Proposition 2.2. *Let P be a* THIN *grid n-ogon with $r = \frac{n-4}{2} \geq 1$ reflex vertices, then $G(P)$ is a path graph (see examples in Fig. 2 (a)).*

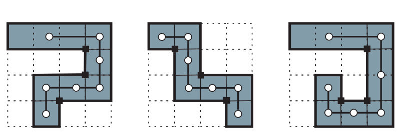


Figure 2: Three THIN grid 10-ogons and respective dual graphs.

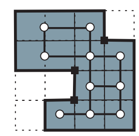


Figure 3: A grid 10-ogon and respective dual graph.

The proof of this proposition is done by induction on r and uses lemma 2.1.

Proposition 2.3. *Let P be a grid n-ogon, with $n > 6$. If P is not* THIN *then $G(P)$ is not a tree (see example in Fig. 3 (b)).*

Proposition 2.2 establishes that, being P a THIN grid n-ogon, $G(P)$ is a path graph. So, each r-piece of $\Pi(P)$ is adjacent to at most two r-pieces. In this way, each r-piece has at most 2 interior edges. Consequently, we conclude that in $\Pi(P)$ there are 3 types of r-pieces: ***Type 1***: with one interior edge and three boundary edges; ***Type 2***: with two interior edges not adjacent and two boundary edges not adjacent; ***Type 3***: with two adjacent interior edges and two adjacent boundary edges. The r-pieces of the Type 1 correspond to leaves of $G(P)$ and those of the Type 2 and Type 3 correspond to nodes of degree 2. We showed that of the 4 vertices of the r-pieces of Type 1 three are vertices of P, being two reflex and the other convex, and the other vertex is an interior point of an edge of P. Of the 4 vertices of the r-pieces of Type 2 two are convex vertices of P and the other two are interior points of edges of P. And finally, of the 4 vertices of the r-pieces of Type 2 two are vertices of P, being one reflex and the other convex, and the other two are interior points of edges of P (see Fig. 4).

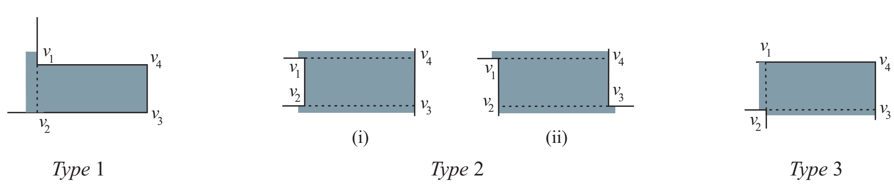


Figure 4: r-pieces of a THIN, from left to right: Type 1, Type 2 and Type 3

Now, we are going to define the *skeleton* of a THIN grid n-ogon. Let P be a THIN grid n-ogon, since $G(P)$ is a path graph, we can say that P has two *"extremes"*: the r-pieces associated with the leafs of the dual graph. We will denote by *kernel* the extreme that has the horizontal edge with highest y-coordinate. From this graph we can obtain an orthogonal polygonal curve (i.e., a polygonal curve with horizontal or vertical edges) in the following way: we take the centroid of each r-piece, then we connect each one with the centroids of the adjacent r-pieces and, finally, we remove the central vertex of each three aligned vertices, as we can see in Fig.5. We choose, for the first vertex of this orthogonal curve the kernel's centroid.

Lemma 2.4. *The skeleton of a* THIN *grid n-ogon is an orthogonal polygonal curve with $r + 2$ vertices.*

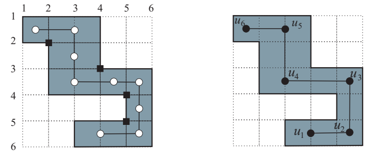


Figure 5: THIN with $r = 4$; on the left is represented its dual graph and on the right its skeleton.

3 Problems related to Thin grid n-ogons

As we saw in section 2, on the contrary of the FATs the THIN grid n-ogons are not unique. In fact, there is 1 THIN 6-ogon, there are 2 THIN 8-ogons, there are 30 THIN 10-ogons, there are 149 THIN 12-ogons, etc. Thus, it is interesting to evidence that the number of THIN grid n-ogons ($|\text{THIN}(n)|$) grows exponentially. Will it exist some expression that relates n to $|\text{THIN}(n)|$? As a step for the resolution of this problem we will first group the THINs into classes. In section 2 we defined the skeleton of a THIN grid n-ogon, now, using this concept, we will group THIN grid n-ogons into classes.

From the skeleton of a THIN grid n-ogon, we can always represent it by a chain of 0's and 1's, with length r. For that we proceed in the following way: we transverse its skeleton, starting at vertex u_1, and then we represent each turn left by 1 and each turn right by 0. For example, the chain that represents the THIN illustrated in Fig.8 is 1101. Now, we will define two operations on these chains: the *complementary operation* and the *inversion operation*.

Definition 3.1. Let c be a chain of 0's and 1's, with length r, i.e., $c = b_1 b_2 \ldots b_r$, where $b_i = 0$ or $b_i = 1$, for $i = 0, 2, \ldots, r$. The complementary operation is an operation which takes c as argument and returns its complementary $c^* = b_1^* b_2^* \ldots b_r^*$, where $b_i^* = 1$ if $b_i = 0$ and $b_i^* = 0$ if $b_i = 1$, for $i = 0, 2, 3, \ldots, r$. The inversion operation is an operation which takes c as argument and returns its inverse $c^{-1} = b_r b_{r-1} \ldots b_2 b_1$. For example, the complementary of the chain $c = 100011$ is the chain $c^* = 011100$ and its inverse is $c^{-1} = 110001$.

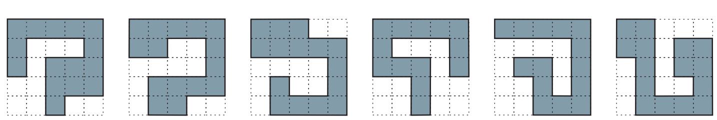
Easily we can check that, $(c^*)^{-1} = (c^{-1})^*$, $(c^*)^* = c$ and $(c^{-1})^{-1} = c$.

Proposition 3.2. *Let $\mathcal{C}_r$ be the set of all chains, of 0's and 1's, with length r. The relation $\sim$ defined on $\mathcal{C}_r$ by* $c_1 \sim c_2 \Leftrightarrow c_2 = c_1 \vee c_2 = c_1^{-1} \vee c_2 = c_1^* \vee c_2 = (c_1^*)^{-1}$ *, is an equivalence relation.*

Consider, now, the quotient set of $\mathcal{C}_r$ by $\sim$, $\mathcal{C}_r/_\sim = \{[c_1]_\sim : c_1 \in \mathcal{C}_r\}$. Note that, each equivalence class has more than one representant. We assume that the representant of each equivalence class always starts by 1.

Proposition 3.3. *Let $\mathcal{P}_r$ be the set of all* THIN *grid n-ogons, with r reflex vertices. The relation $\equiv$ defined on $\mathcal{P}_r$ by* $P_1 \equiv P_2 \Leftrightarrow c_1 \sim c_2$*, where c_1 and c_2 are the chains that represent P_1 and P_2, respectively , is an equivalence relation.*

The proof of this proposition is trivial. Consider $\mathcal{P}_r/_\equiv = \{[P_1]_\equiv : P_1 \in C_r\}$. Let $P_1, P_2 \in \mathcal{P}_r$ and $c_1, c_2 \in \mathcal{C}_r$ the chains that represent them, respectively. Notice that, P_1 and P_2 belong to the same class (i.e., P_1 and P_2 are *equivalents*) if one of the following conditions is true: (i) $c_1 = c_2$; (ii) $c_2 = c_1^{-1}$; (iii) $c_2 = c_1^*$ or (iv) $c_2 = (c_1^*)^{-1}$. Observe that, geometrically, (ii) can correspond to an horizontal reflection and (iii) to a vertical reflection. In Fig. 6 are illustrated six THINs with 4 reflex vertices that belong to the same class.



0111 0111 1110 1000 0111 0001

Figure 6: THINs with 4 reflex vertices and respective chains

We already know, that each THIN grid n-ogon with r reflex vertices, can be represented by a chain of 0's and 1's, with length r. Now, we can place the following question: Let c be chain of 0's and 1's with length r, started by 1. It is always possible to construct a THIN, with r reflex vertices, whose chain that represents it is c? To answer this question we present the next algorithm:

Algorithm 3.4. Let c be a chain of 0's and 1's, of length r, started by 1.

1. From the chain draw a skeleton ignoring collinearities.
2. Make an horizontal sweep, from left to right, to eliminate vertical collinearities. This elimination is made modifying the edge corresponding to the beginning of the polygon. If two edges correspond to the beginning of the polygon, or no edge correspond to the beginning of the polygon, is indifferent the one that is modified.
3. Repeat the previous step until being without collinear vertical edges.
4. Make vertical sweep, from bottom to top, to eliminate horizontal collinearities. This elimination is made modifying the edge corresponding to the beginning of the polygon. If two edges correspond to the beginning of the polygon, or no edge correspond to the beginning of the polygon, is indifferent the one that is modified.
5. Repeat the previous step until being without collinear horizontal edges.

Figure 7(a) illustrate this algorithm.

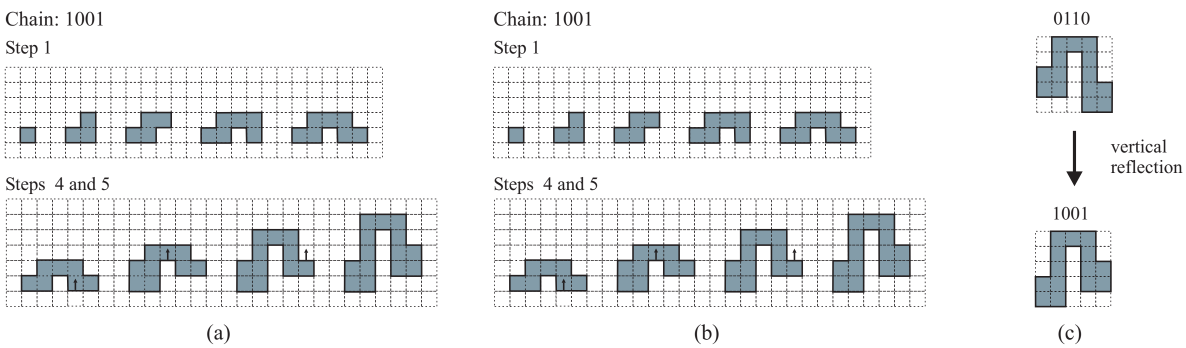


Figure 7: (a) Illustration of the algorithm, $c = 1001$; (b) Illustration of the algorithm, $c = 1001$; (c) The chain that represents the THIN after the reflection is $c = 1001$.

Observe that: In step 1 if we draw a skeleton only ignoring the collinearities, we can get a skeleton that does not correspond to the given chain. In step 2., when the polygon has beginning in two collinear edges, nor always is indifferent the choice of the edge, for example, in Fig. 7 (b), the chain that represents the constructed THIN is 0110 (complementary of the given chain). However, this algorithm always generates a THIN grid n-ogon whose chain that represents it is equivalent to c. Thus, if the chain that represents the THIN, generated by this algorithm, is c^*, c^{-1} ou $(c^*)^{-1}$, is enough to make a vertical reflection, a horizontal reflection or a vertical reflection followed by a horizontal reflection, respectively, so that the chain that represents it is exactly c, see Fig. 7 (c) for illustration. Thus, this algorithm proves that for each chain of 0's and 1's, with length r, started by 1, there is a THIN grid n-ogon with r reflex vertices represented by it.

Now, we are going to count the number of classes of THIN grid n-ogons with r reflex vertices. To solve this problem we, first, proved the following proposition:

Proposition 3.5. *The correspondence $f : \mathcal{P}_r/_{\equiv} \to \mathcal{C}_r/_{\sim}$ defined by $f([P_1]) = [c_1]$, where $c_1 \in \mathcal{C}_r$ is the chain that represents $P_1 \in \mathcal{P}_r$, which is a representant of the class $[P_1]$, is a bijective function.*

Proposition 3.6. *The number of classes of THIN grid n-ogons with r reflex ($r \geq 2$) is equal to $2^{r-2} + 2^{\frac{1}{2}(r-3)}$, if r is odd and $2^{r-2} + 2^{\frac{1}{2}(r-2)}$, if r is even.*

Proof. By proposition 3.5 we can conclude that $|\mathcal{P}_r/_{\equiv}| = |\mathcal{C}_r/_{\sim}|$, so we just have to calculate $|C_r/_{\sim}|$. The cardinal of C_r is 2^r and the number of symmetrical chains ($c = c^{-1}$), with length r, is $2^{\lceil \frac{r}{2} \rceil}$. If a chain c is symmetrical, then its equivalence class is constituted by two chains, c e c^*. If a chain c is not symmetrical, to find the cardinal of its class, we have to distinguish two cases: r odd and r even. If r is odd, all the chains have 4 equivalent chains: c, c^{-1}, c^* e $(c^*)^{-1}$ (for example: $c = 11010, 01011, 00101$ and 10100). If r is even, there are chains that have 4 equivalent chains (e.g., $c = 1110$) and chains that only have 2 equivalent chains, this case happens when $c^* = c^{-1}$ (e.g., for the chain $c = 1100$, $c^* = c^{-1} = 0011$).

Let us count, now, the number of equivalence classes. If r is odd, the number of equivalence classes of symmetrical chains is $\frac{1}{2}\#(\text{symmetrical chains}) = \frac{1}{2}\ 2^{\frac{r+1}{2}} = 2^{\frac{r-1}{2}}$ and the number of equivalence classes of not symmetrical chains is $\frac{1}{4}\#(\text{not symmetrical chains}) = \frac{1}{4}\ (2^r - 2^{\frac{r+1}{2}}) = 2^{r-2} - 2^{\frac{r-3}{2}}$. Thus, if r is odd, in the total, the number of equivalence classes is $2^{r-2} + 2^{\frac{r-1}{2}} - 2^{\frac{r-3}{2}} = 2^{r-2} + 2^{\frac{1}{2}(r-3)}$.

If r is even, the number of equivalence classes of symmetrical chains is $\frac{1}{2}\#(\text{symmetrical chains}) = \frac{1}{2}\ 2^{\frac{r}{2}} = 2^{\frac{1}{2}(r-2)}$. The number of equivalence classes of not symmetrical chains constituted by two chains (for example, the classes of the chains 101010, 1100, 110100,...) is $\frac{1}{2}\ 2^{\frac{r}{2}} = 2^{\frac{1}{2}(r-2)}$. In fact, to obtain $c^* = c^{-1}$, the second half of the chain is completely determined by the first half. Therefore, the cardinal of these classes is half of the number of chains of this type. And, the number of equivalence classes of not symmetrical chains constituted by four chains is $\frac{1}{4}\#(\text{All} - \text{Symmetric} - (\text{Chains with } c^* = c^{-1})) = \frac{1}{4}\ (2^r - 2^{\frac{r}{2}} - 2^{\frac{r}{2}}) = 2^{r-2} - 2^{\frac{1}{2}(r-2)}$. Thus, if r is even, in the total, the number of equivalence classes is $2^{r-2} + 2^{\frac{1}{2}(r-2)}$.

□

However, still there are some open problems to solve, such as: How many elements Thin grid n-ogons have each class? It will be possible to find an algorithm that generate all Thin grid n-ogons of the same class?

Note that, solving the first problem we also solve the initial problem, that is: will it exist some expression that relates n to $|\text{Thin}(n)|$?

4 Visibility Problems on grid n-ogons

Of the problems related to grid n-ogons, the guarding and hiding problems are the ones that motivate us more, particulary the MVG and MHVS problems. Since Thin and Fat n-ogons are the classes for which the number of r-pieces is minimum and maximum, we think that they can be representative of extremal behavior, besides they are used experimentally to evaluate approximate methods of resolution of the MVG problem, so we started with them. We already proved that to guard any Fat grid n-ogon it is always sufficient two $\frac{\pi}{2}$ vertex guards (*vertex guards with* $\frac{\pi}{2}$ *range visibility*) and established where they must be placed [4]. However, Thin grid n-ogons are much more difficult to guard, in spite of their having much fewer r-pieces than Fats. Besides, they are not unique, so we tried to characterize structural properties of classes of Thins that allow to simplify the problem's study. Up to now the only quite characterized subclasses are the Min-Area and the Spiral grid n-ogons. We proved that to guard any Min-Area and Spiral grid n-ogon are necessary $\lceil \frac{n}{6} \rceil$ and $\lfloor \frac{n}{4} \rfloor$ vertex guards, respectively. Moreover, we showed where those guards could be placed [5, 6]. In the next section we will study the Maximum Hidden Vertex Set Problem on Thin grid n-ogons.

4.1 Maximum Hidden Vertex Set Problem on Thin grid n-ogons

Given a simple polygon, P, and a set of vertices of P, HV, we say that HV is an *hidden vertex set* if no two vertices in HV see each other. The Maximum Hidden Vertex Set problem on a simple polygon asks for an hidden vertex set, HV, of maximum cardinality. We will call the elements of HV *hidden vertices*. Shermer [7] proved that the size of the maximum hidden vertex set of a n-ogon is at most $\frac{n-2}{2}$, this tight bound is achieved in staircase polygons. We will show that, given a Thin grid n-ogon the maximum cardinality of an hidden vertex set is $\lceil \frac{n}{4} \rceil$.

Let P be a THIN grid n-ogon and $S = u_1 u_2 \dots u_{\frac{n}{2}}$ its skeleton. Let us assume, without loss of generality, that the first edge of S, $\overline{u_1 u_2}$, is horizontal and that u_2 is to the right of u_1. Notices that, the boundary of P consists of two joined polygonal chains, c_1 and c_2, "parallels" to S, where the first edge of c_1 is a bottom edge and the first edge of c_2 is a top edge. Notices that, c_1 and c_2 can be expressed as ordered sequences of vertices $c_1 = v_1^1 v_2^1 \dots v_{\frac{n}{2}}^1$ and $c_2 = v_1^2 v_2^2 \dots v_{\frac{n}{2}}^2$, where v_i^1 denotes the i'esim vertex of c_1 and v_i^2 denotes the i'esim vertex of c_2 (see Fig.8). In this way, $BND(P) = c_1 \cup \overline{v_{\frac{n}{2}}^1 v_{\frac{n}{2}}^2} \cup c_2 \cup \overline{v_1^2 v_1^1}$. Observe, also, that, if we transverse S, starting at vertex u_1, c_1 it is always on the right of S and c_2 on the left.

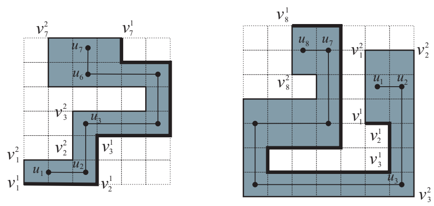


Figure 8: Two THIN grid n-ogons, its skeletons and the chains c_1 and c_2 (chain c_1 is in bold).

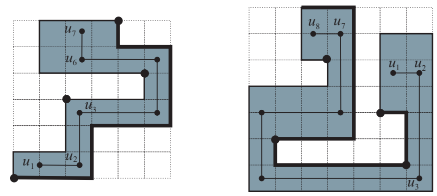


Figure 9: Two THIN grid n-ogons and marked hidden vertices (chain c_1 is in bold).

To each vertex of the skeleton we make to correspond two vertices of the polygon, one in c_1 and another one in c_2. That is, to $u_i \in S$, we make to correspond the vertices $v_i^1 \in c_1$ and $v_i^2 \in c_2$. And to each edge of the skeleton we make to correspond two parallel edges of the polygon, one in c_1 and another one in c_2. That is, to $\overline{u_i u_{i+1}} \in S$, we make to correspond the edges $\overline{v_i^1 v_{i+1}^1} \in c_1$ and $\overline{v_i^2 v_{i+1}^2} \in c_2$. Notices that, by construction of the skeleton, we can easily see that any point of $\overline{v_i^1 v_{i+1}^1}$ sees any point of $\overline{v_i^2 v_{i+1}^2}$.

Now, for each $u_{2k-1} \in S$ with $k = 1, \dots, \lceil \frac{n}{4} \rceil$, we mark an hidden vertex in P, in the following way: for $k = 1$ we mark v_1^1; for $k \neq 1$ we mark v_{2k-1}^1 or v_{2k-1}^2, depending if v_{2k-2}^1 is reflex or convex, respectively (see Fig.9, for illustration).

Notices that the $\lceil \frac{n}{4} \rceil$ marked vertices form an hidden vertex set, since each time that we mark a new vertex as hidden we can guarantee that it does not see none of the vertices that previously had been marked as hidden. In fact, for $k = 1$ it is trivial. For $k \neq 1$, we have two cases, depending if v_{2k-2}^1 is reflex (*Case* 1) or convex (*Case* 2), as we can se in Fig.10.

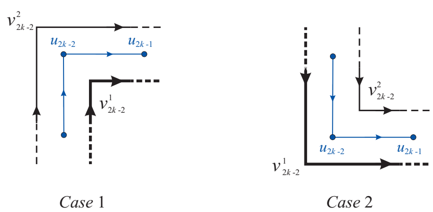


Figure 10: On the left v_{2k-2}^1 is reflex and on the right is convex.

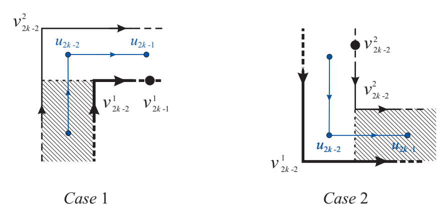


Figure 11: The shaded zone is not seen by the marked vertex.

In case 1 the vertex that is marked as hidden is the vertex v_{2k-1}^1 and in case 2 is the vertex v_{2k-1}^2. In both cases the marked vertex does not see none of already marked as hidden, since of the already "visited" vertices this one only sees v_{2k-2}^1 and v_{2k-2}^2, such as it is possible to check easily in Fig.11. Observe that the choice of another vertex does not guarantee us this, because in any case the other vertex sees more backwards. Therefore, lemma 4.1 follows.

Lemma 4.1. *For any* THIN *grid n-ogon there is an hidden vertex set HV and* $|HV| = \lceil \frac{n}{4} \rceil$.

Now we will prove that the maximum cardinality of an hidden vertex set in a THIN grid n-ogon is $\lceil \frac{n}{4} \rceil$. To prove this result we introduce lemma 4.2.

Lemma 4.2. *Let P be a* THIN *grid n-ogon and S its skeleton. To two consecutive vertices in S do not correspond hidden vertices in P.*

Proof. Let u_i and u_{i+1} be two consecutive vertices in S. The corresponding vertices in P are v_i^1, v_i^2, v_{i+1}^1 and v_{i+1}^2, respectively. By the correspondence previously established, any point of the edge $\overline{v_i^1 v_{i+1}^1}$ sees any point of the edge $\overline{v_i^2 v_{i+1}^2}$, in particular the vertices of the edges. Therefore, v_i^1 sees v_i^2 and v_{i+1}^2; and v_{i+1}^1 sees v_i^2 and v_{i+1}^2. And it is obvious, that v_i^1 sees v_{i+1}^1 and that v_i^2 sees v_{i+1}^2.

□

Theorem 4.3. *Let P be a* Thin *grid n-ogon. The maximum cardinality of an hidden vertex set in P is $\lceil \frac{n}{4} \rceil$.*

Proof. By lemma 4.1 there is an hidden vertex set in P with cardinality $\lceil \frac{n}{4} \rceil$. Suppose, now, that there is an hidden vertex set HV, with $|HV| \geq \lceil \frac{n}{4} \rceil + 1$. Since the skeleton of P has $\lceil \frac{n}{4} \rceil$ vertices with index odd, this implies that an hidden vertex will have that to be placed in a vertex in P that corresponds to a vertex of the skeleton with index even. In other words, it means that two hidden vertices will have to be placed in two vertices of P that correspond to two consecutive vertices of the skeleton, in contradiction with lemma 4.2. □

5 Conclusions

We defined a particular type of polygons - grid n-ogons - and presented some results related to them. Of the hiding problems related to the grid n-ogons, it is the MHVS problem that motivates us more. We proved that the maximum cardinality of an hidden vertex set in a Thin grid n-ogon is $\lceil \frac{n}{4} \rceil$. Moreover, we established a possible positioning for those hidden vertices. We also established a possible classification for Thin grid n-ogon, as a step to launch an expression that relates n to $|\text{Thin}(n)|$.

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