New results on lower bounds for the number of ($\leq k$)-facets *

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Abstract

In this paper we present three different results dealing with the number of ($\leq k$)-facets of a set of points:

1. We give structural properties of sets in the plane that achieve the optimal lower bound $3\binom{k+2}{2}$ of ($\leq k$)-edges for a fixed $k \leq \lfloor n/3 \rfloor - 1$;
2. We show that the new lower bound $3\binom{k+2}{2} + 3\binom{k-\lfloor\frac{n}{3}\rfloor+2}{2}$ for the number of ($\leq k$)-edges of a planar point set shown in [2] is optimal in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$;
3. We show that, for $k < n/4$ the number of ($\leq k$)-facets a set of n points in $\mathbb{R}^3$ in general position is at least $4\binom{k+3}{3}$, and that this bound is tight in that range.

1 Introduction

In this paper we deal with the problem of giving lower bounds to the number of ($\leq k$)-facets of a set of points S: An oriented simplex with vertices at points of S is said to be a j-*facet* of S if it has exactly j points in the positive side of its affine hull. Similarly, the simplex is said to be an ($\leq k$)-*facet* if it has at most k points in the positive side of its affine hull.

The number of j-facets of S is denoted by $e_j(S)$ and $E_k(S) = \sum_{j=0}^{k} e_j(S)$ is the number of ($\leq k$)-facets (the set S can be omitted if it is clear from the context). Giving bounds on these quantities, and on the number of the companion concept of k-set, is one of the central problems in Discrete and Computational Geometry, and has a long history that we will not try to summarize here. Chapter 8.3 in [5] is a complete and up to date survey of results and open problems in the area.

Regarding lower bounds for $E_k(S)$, which is the main topic of this paper, the problem was first studied by Edelsbrunner et al. [7] due to its connections with the complexity of higher order Voronoi diagrams. In that paper it was stated that $E_k(S) \geq 3\binom{k+2}{2}$ and an example was given showing that the bound is tight if $k \leq \lfloor n/3 \rfloor - 1$. Unfortunately, the proof of the bound was not correct and a correct proof, based on circular sequences, was independently found by Abrego and Fernández-Merchant [1] and Lovász et al. [8], where the problem was revisited due to its strong connection with the rectilinear crossing number of the complete graph or, equivalently, with the number of convex quadrilaterals in a set of points.

This lower bound was slightly improved for $k \geq \lfloor \frac{n}{3} \rfloor$ by Balogh and Salazar in [4], again using circular sequences. Recently, and based on the observation that it suffices to proof the bound for sets

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with triangular convex hull, we have shown [2] that

$$E_k(S) \geq 3\binom{k+2}{2} + \sum_{j=\lfloor \frac{n}{3} \rfloor}^{k} (3j - n + 3). \tag{1}$$

In this paper we deal with three different problems related to lower bounds for $E_k(S)$: In Section 2, we study structural properties of sets in the plane that achieve the lower bound $3\binom{k+2}{2}$ for a fixed $k \leq \lfloor n/3 \rfloor - 1$. The main result of this section is that if $E_k(S)$ is minimum for such a k, then $E_j(S)$ is also minimum for every $0 \leq j < k$. In Section 3 we give a construction which shows that the lower bound in Equation (1) is optimal in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$. Finally, in Section 4 we study the 3-dimensional version of the problem and show that, for $k < n/4$, the number of $(\leq k)$-facets of a set of n points in general position in $\mathbb{R}^3$ is at least $4\binom{k+3}{3}$, and that this bound is tight in that range. The proof of this result is based on the fact that, similarly to the planar case, it is sufficient to prove the lower bound for sets with four vertices in the convex hull.

2 Optimal $(\leq k)$-set vectors

Let us denote by $\mathcal{E}_k(S)$ the set of all $(\leq k)$-edges of S, that is $E_k(S)$ is the cardinality of $\mathcal{E}_k(S)$. We recall that for fixed $k \leq \lfloor \frac{n}{3} \rfloor - 1$, $E_k(S)$ optimal means $E_k(S) = 3\binom{k+2}{2}$. The *positive side* of a j-edge, $j < \frac{n-2}{2}$, is the open half plane of its supporting line with j vertices of S in it.

Let Δ be a triangle spanned by three vertices of the convex hull of S. An edge e of Δ is called *good* if the open half plane of its supporting line which contains the third vertex of Δ, contains at least $\frac{n-2}{2}$ points from S. Δ is called *good* if it consists of three good edges.

Lemma 2.1. *Any set S of n points contains a good triangle spanned by vertices of its convex hull.*

Proof. Let Δ be an arbitrary triangle spanned by vertices of the convex hull of S. Assume that Δ is not good. Then observe that only one edge e of Δ is not good and let v be the vertex of Δ not incident to e. Choose a point v' of the convex hull of S opposite to v with respect to e. Then e and v' induce a triangle Δ' in which e is a good edge. If Δ' is a good triangle we are done. Otherwise we iterate this process. As the subset of vertices of S we consider is strictly decreasing (restricted by the half plane induced by e), this process terminates with a good triangle. $\square$

Lemma 2.2. *Let $a, b, c \in S$ be the vertices of a good triangle Δ of S and let $S' = S \setminus \{a, b, c\}$. Then*

$$E_k(S) \geq E_{k-2}(S') + 3 + 6k.$$

Proof. Consider an $(\leq k-2)$-edge e of S'. If e intersects Δ then it is an $(\leq k)$-edge of S as it can have at most two vertices of Δ on its positive side. If e misses Δ then it can not have Δ on its positive side and it is thus an $(\leq k-2)$-edge of S. Thus any edge of $\mathcal{E}_{k-2}(S')$ also belongs to $\mathcal{E}_k(S)$. In addition there are $2(k+1)$ $(\leq k)$-edges incident to each of the convex hull vertices a, b, and c. At most three edges might be incident to two of these vertices (the edges of Δ) and the bound follows. $\square$

Corollary 2.3. *Let $a, b, c \in S$ be the vertices of a good triangle of S and let $S' = S \setminus \{a, b, c\}$. If $E_k(S)$ is optimal then $E_{k-2}(S')$ is optimal and $E_k(S) = E_{k-2}(S') + 3 + 6k$.*

Proof. From the previous lemma we have $3\binom{k+2}{2} = E_k(S) \geq E_{k-2}(S') + 3 + 6k \geq 3\binom{k}{2} + 3 + 6k = 3\binom{k+2}{2}$. Thus for both greater-or-equal signs equality holds and therefore $E_{k-2}(S')$ has to be optimal. $\square$

Theorem 2.4. *If $E_k(S) = 3\binom{k+2}{2}$, then S has a triangular convex hull.*

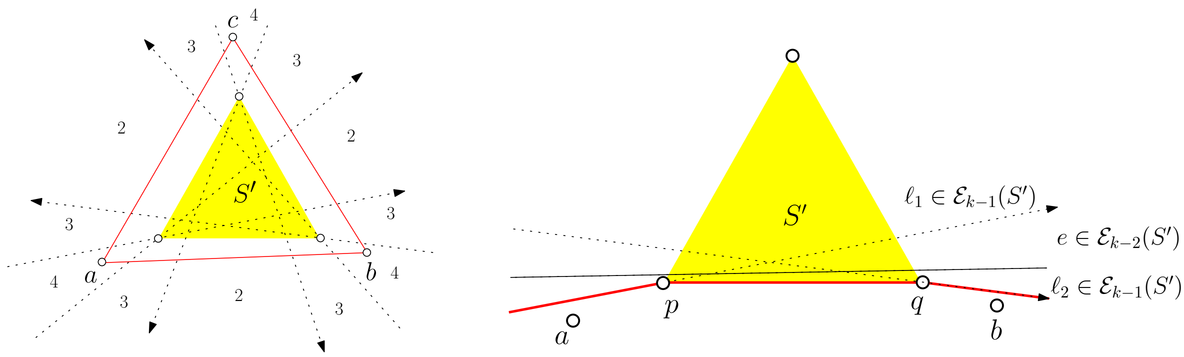

Figure 1: Left: Each $(k-1)$-edge of S' incident to a convex hull vertex of S' (supporting lines are shown as dashed lines) has two vertices of Δ on its positive side. Right: All $(k-2)$-edges of S' (supporting lines are shown as dotted lines) lie above the (bold) lower envelope.

Proof. We prove the statement by induction over k. For $k = 0$ nothing has to be proven, so let $k = 1$ and let $h = |CH(S)|$. Optimality of $E_1(S)$ means $E_1(S) = 9$. We have h 0-edges and at least h 1-edges (two per convex hull vertex, but each edge might be counted twice). Thus $E_1(S) = 9 \geq 2h$ and therefore $h \leq 4$. Assume now $h = 4$. Then at most two 1-edges can be counted twice, namely the two diagonals of the convex hull. Thus we have $4 + 8 - 2 = 10$ (≤ 1)-edges and we conclude that, for optimal $E_1(S)$, S has a triangular convex hull.

For the general case let $k \geq 2$ and define Δ and S' as in Lemma 2.2. From Corollary 2.3 we know that $E_{k-2}(S')$ is optimal and by induction the convex hull of S' is a triangle. Moreover, Corollary 2.3 implies that no $(k-1)$-edge of S' can be an $(\leq k)$-edge of S. Thus any $(k-1)$-edge of S' must have two vertices of Δ on its positive side. Consider the six $(k-1)$-edges of S' incident to the three convex hull vertices of S', see Figure 1 (left). There the supporting lines of these $(k-1)$-edges are drawn as dashed lines and S' is depicted as the central triangle. Each cell outside of S' in the arrangement of the supporting lines contains a number counting those of the $(k-1)$-edges considered which have this cell on their positive side. A simple counting argument shows that the only way of placing the three vertices a, b, c of Δ such that each $(k-1)$-edge of S' drawn has two vertices on its positive side is to place one in each cell labelled with a 4. We conclude that no vertex of S' can be on the convex hull of S and the theorem follows. $\square$

Corollary 2.5. *If $E_k(S) = 3\binom{k+2}{2}$, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles.*

Proof. From the optimality of $E_k(S)$ it follows by Corollary 2.3 that we can iteratively remove the outermost $\lceil \frac{k}{2} \rceil$ layers to obtain optimal subsets, which, by Theorem 2.4, have triangular convex hulls. $\square$

Lemma 2.6. *A j-edge, $j \geq 1$, of S either has two vertices of the convex hull of S on its positive side, or is incident to one of those vertices and has another on its positive side.*

Proof. From Corollary 2.3 we have that $E_j(S) = E_{j-2}(S') + 3 + 6k$. Thus an j-edge of S either stems from a $(j-2)$-edge of S' and then has two vertices of the convex hull of S on its positive side, or it comes from the $3 + 6k$ edges incident to a convex hull vertex of S. $\square$

Note that we have not proven that all $(j-2)$-edges of S' need to be j-edges of S. The above lemma only states that those which in fact are, have the claimed structure. Anyway, the next result shows that for optimal $E_k(S)$ all edges have this structure.

Theorem 2.7. *Let $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$. If $E_k(S)$ is optimal, then $E_j(S)$, $0 \leq j \leq k$, is optimal.*

Proof. We prove the theorem by induction on k. For $k = 0, 1$ the theorem is equivalent to Theorem 2.4, so let $k \geq 2$. It is sufficient to show that optimality of $E_k(S)$ implies optimality of $E_{k-1}(S)$, as the theorem follows by induction. Similar to Lemma 2.2 we have

$$E_{k-1}(S) \geq E_{k-3}(S') + 3 + 6(k-1) = 3\binom{k+1}{2}.$$

The last equality follows from the fact that $E_{k-2}(S')$ is optimal by Corollary 2.3 and implies optimality of $E_{k-3}(S')$ by induction. To prove optimality of $E_{k-1}(S)$ it thus remains to show that no $(k-1)$-edge and no $(k-2)$-edge of S' can be an $(\leq k-1)$-edge of S. For $(k-1)$-edges of S' this is obviously true, as they have at least one vertex of the convex hull of S on their positive side.

So let e be an $(k-2)$-edge of S' and let p and q be the vertices of the convex hull of S' incident to e or on the positive side of e. The existence of p and q is guaranteed by Lemma 2.6. Without loss of generality, assume that the edge pq is horizontal with the remaining vertices of S' above it, see Figure 1 (right) for the rest of the proof. Let ℓ_1 be the $(k-1)$-edge of S' incident to p which has q on its positive side and ℓ_2 the $(k-1)$-edge incident to q and having p on its positive side. The *boundary chain* is the lower envelope of ℓ_1, pq, and ℓ_2. We claim that e does not intersect the boundary chain and lies above it. If e is incident to p or q then the claim is obviously true. Otherwise observe that e has to intersect the supporting lines of both considered $(k-1)$-edges in the interior of S', as otherwise there would be too many vertices on the positive side of e. But then again e lies above the boundary chain and the claim follows.

From the proof of Theorem 2.4 we know that two of the vertices of the convex hull of S have to lie below our boundary chain (below the $(k-1)$-edges, see a and b in Figure 1, right) and thus on the positive side of e. Therefore e has at least k vertices of S on its positive side and does not belong to $\mathcal{E}_{k-1}(S)$. We conclude that $E_{k-1}(S)$ is optimal and the theorem follows. □

3 Tightness of the lower bound for $(\leq k)$-edges in $\mathbb{R}^2$

In this section we show a point configuration which proves that the lower bound for $E_k(S)$ given in [2] is tight for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$. This solves an open conjecture in [2]. Consider the configuration in Figure 2 (left), which is composed of three rotationally symmetric chains, each one associated to a convex hull vertex, fulfilling the following properties (where left and right are considered with respect to the corresponding convex hull vertex):

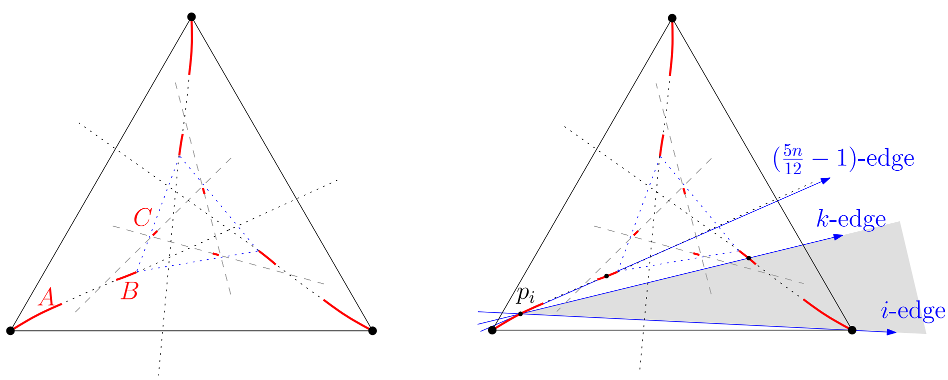

Figure 2: Left: Configuration for which the bound in [2] for $E_k(S)$ is tight. Right: For $i \in \{0, \ldots, \frac{2n}{12} - 1\}$, exactly one j-edge appears for each $j \in \{i, \ldots, k\}$.

- The first part of the chain is slightly convex to the right and contains $\frac{3n}{12}$ points, with a hole between the first $\frac{2n}{12}$ points (which we call subchain A) and the remaining $\frac{n}{12}$ points (called subchain B).
- Each chain is completed with a subchain C, composed of another $\frac{n}{12}$ points slightly convex to the left.

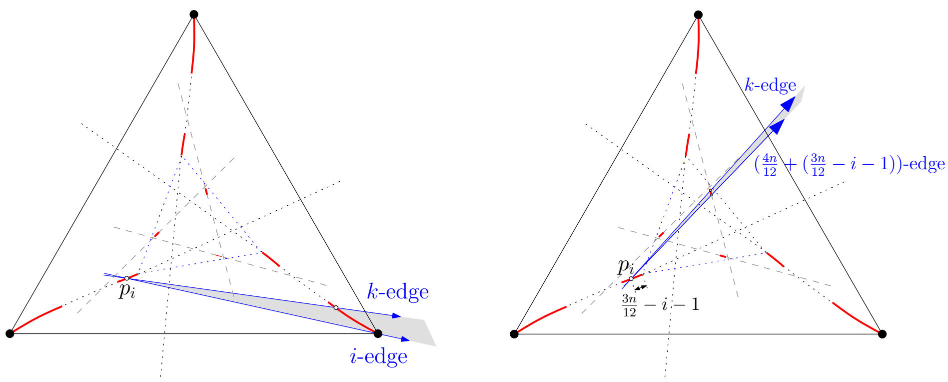

Figure 3: Left: For $i \in \{\frac{2n}{12}, \ldots, \frac{3n}{12}-1\}$, exactly one j-edge appears for each $j \in \{i, \ldots, k\}$. Right: For $i \in \{\frac{2n}{12}, \ldots, \frac{3n}{12}-1\}$, exactly one j-edge appears for each $j \in \{\frac{7n}{12}-i-1, \ldots, k\}$.

- All the lines spanned by two points in $A \cup B$ leave to the right the next chain in counterclockwise order, and to the left both the points in C and those in the remaining chain.
- All the lines spanned by two points in C separate subchains A and B. Furthermore, they leave to the right both other subchains of type C.
- The triangle defined by the innermost points of chains of type B contains all the chains of type C.

Theorem 3.1. *For the point configuration S defined above and $\frac{n}{3} \le k \le \frac{5n}{12}-1$,*

$$E_k(S) = 3\binom{k+2}{2} + 3\binom{k-\frac{n}{3}+2}{2}.$$

Proof. Because of the rotational symmetry, we can focus on one of the three chains $A \cup B \cup C$ and let p_i be the $(i+1)$-th point on that chain. We will count oriented j-edges of type $\overrightarrow{p_i q}$ (i.e. with p_i on the tail) for $j \le k$. In order to do so we rotate counterclockwise a ray based on p_i, starting from the one passing through the convex hull vertex of the next chain in counterclockwise order. Three cases arise, depending on the index i of p_i, which correspond to p_i lying on one of the three subchains:

(A) For $i \in \{0, \ldots, \frac{2n}{12}-1\}$, exactly one j-edge appears for each j in the range $j \in \{i, \ldots, k\}$, while all the remaining j-edges $\overrightarrow{p_i q}$ in the rotation have $j > k$ since at some point we find a $(\frac{5n}{12}-1)$-edge and after that all the j-edges found have $j \ge \frac{5n}{12} > k$. See Figure 2 (right).

(B) For $i \in \{\frac{2n}{12}, \ldots, \frac{3n}{12}-1\}$, exactly one j-edge appears for each j in the ranges $j \in \{i, \ldots, k\}$ and $j \in \{\frac{7n}{12}-i-1, \ldots, k\}$, while all the remaining j-edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 3.

(C) For $i \in \{\frac{3n}{12}, \ldots, \frac{4n}{12}-1\}$, exactly one j-edge appears for each j in the ranges $j \in \{i, \ldots, k\}$ and $j \in \{\frac{8n}{12}-i-1, \ldots, k\}$, while all the remaining j-edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 4.

Let us point out that, depending on the values of k and i, some of the above ranges could actually be empty. Now we are ready to count the total number of $(\le k)$-edges incident to points p_i on the first chain, which is:

$$\sum_{i=0}^{\frac{2n}{12}-1}(k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1}(k-i+1) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1}(k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1}(k-\frac{7n}{12}+i+2) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1}(k-\frac{8n}{12}+i+2),$$

where the first three summands come from the first ranges of the three cases above, while the two remaining summands come from the second ranges in cases (B) and (C). Merging the first three summands and rewriting the two latter ones, the above sum equals

$$\sum_{i=0}^{\frac{4n}{12}-1}(k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}}(k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}}(k-i+1) = \sum_{j=1}^{k+1} j + \sum_{j=1}^{k-\frac{4n}{12}+1} j = \binom{k+2}{2} + \binom{k-\frac{n}{3}+2}{2},$$

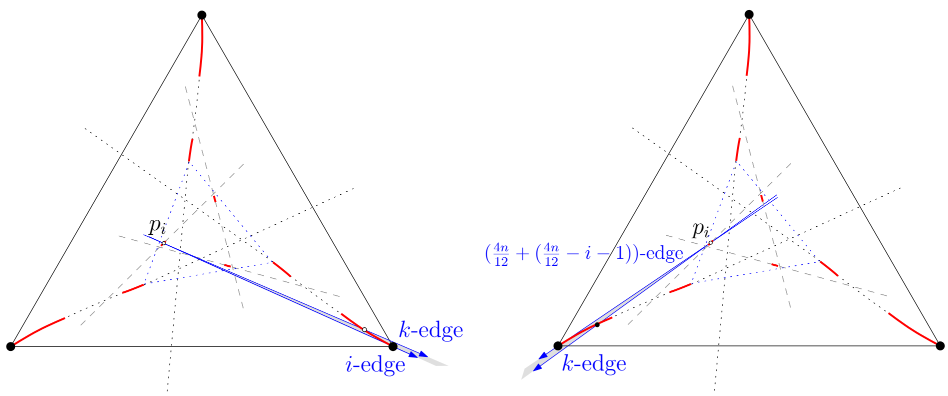

Figure 4: Left: For $i \in \{\frac{3n}{12}, \ldots, \frac{4n}{12} - 1\}$, exactly one j-edge appears for each $j \in \{i, \ldots, k\}$. Right:For $i \in \{\frac{3n}{12}, \ldots, \frac{4n}{12} - 1\}$, exactly one j-edge appears for each $j \in \{\frac{8n}{12} - i - 1, \ldots, k\}$.

where the first equality comes from neglecting the negative summands, due to the above mentioned empty ranges, and merging the first two sums. This result has to be multiplied by the three chains of the configuration, so we get

$$E_k(S) = 3\binom{k+2}{2} + 3\binom{k - \frac{n}{3} + 2}{2},$$

which matches the lower bound stated in [2]. □

4 A lower bound for $(\leq k)$-facets in $\mathbb{R}^3$

Throughout this section, $S \subset \mathbb{R}^3$ will be a set of n points in general position. Given $p, q, r \in S$, we say that the triangle pqr is a *j-facet* of S if the plane containing it partitions S into two subsets with cardinality j and $n - 3 - j$ (we consider unoriented j-facets and, therefore, $j \leq \frac{n-3}{2}$).

Let us recall that $e_k(S)$ and $E_k(S)$ denote, respectively, the number of k-facets and the number of $(\leq k)$-facets of S. The main result of this section is a lower bound for the number of $(\leq k)$-facets of a set of n points in general position in $\mathbb{R}^3$ in the range $0 \leq k \leq \lfloor \frac{n}{4} \rfloor - 1$.

Motivated by the standard definition of convex position, we say that a set of points is in *simplicial position* if its convex hull is a simplex. Our first step is showing that sets minimizing the number of $(\leq k)$-facets (for $0 \leq k \leq \lfloor \frac{n}{4} \rfloor - 1$) are in simplicial position. We use continuous motion and study the events when the number of j-facets changes. These events, called *mutations*, have been previously considered by Andrzejak et al. [3]: p_1 is moving in a continuous way and a mutation occurs when p_1 becomes coplanar with three other points, p_2, p_3, and p_4 in such a way that the orientation of the 4-tuple changes at that instant. We assume that no other 4-tuple of points changes its orientation during this process (see [3] for a more formal definition).

There are two types of mutations in $\mathbb{R}^3$: when points involved in the mutation are coplanar, they can be in convex position or not. The first type of mutation is called a *convex mutation* and the second one a *mutation through triangle* (see Figure 5). We denote by $M(p_1, p_2, p_3, p_4)$ the mutation involving points p_1, p_2, p_3, and p_4. If M is a mutation through triangle, we assume that p_1 crosses the triangle $p_2p_3p_4$ during the mutation. We denote by $-M$ the mutation opposite to M.

Given $j \leq \frac{n-4}{2}$, we say that the mutation $M(p_1, p_2, p_3, p_4)$ has *index* j if there are j points of S (besides p_1) in the halfspace defined by $p_2p_3p_4$ and containing p_1. Observe that if $j < \frac{n-4}{2}$, then $p_2p_3p_4$ is a $(j+1)$-facet and p_1 is contained in its defining halfspace, while if $j = \frac{n-4}{2}$ then $p_2p_3p_4$ is a $\frac{n-4}{2}$-facet, both before and after the mutation (p_1 is not contained in its defining halfspace).

The changes in the j-facet vector produced during a mutation have been previously considered in [3]. In the next lemma we state the result for future reference.

Lemma 4.1 ([3]). *Let S' be the set obtained from S after a mutation M with index j. Then:*

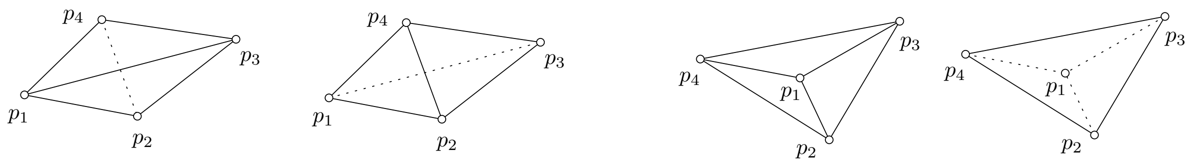

Figure 5: Convex mutation (left) and mutation through triangle (right).

a) If M is a convex mutation, then $e_k(S') = e_k(S)$ for all $0 \leq k \leq \frac{n-3}{2}$.

b) If M is a mutation through triangle then: If $j = \frac{n-4}{2}$ then $e_k(S') = e_k(S)$ for all $0 \leq k \leq \frac{n-3}{2}$. If $j < \frac{n-4}{2}$ then $e_j(S') = e_j(S) - 2$, $e_{j+1}(S') = e_{j+1}(S) + 2$ and $e_k(S') = e_k(S)$ for the rest of the indices.

In order to show that a set of points can be continuously transformed to simplicial position while decreasing the vector of ($\leq k$)-facets we need the concept of *centerpoint* (see [6]). Because we need a set of centerpoints with non-empty interior, we relax the standard definition a little bit: We say that a point $c \in \mathbb{R}^3$ is a centerpoint of S if every closed halfspace containing c contains at least $\lfloor \frac{n}{4} \rfloor$ points of S .

Lemma 4.2. *Let S be a set of points in general position. The set of centerpoints is a polyhedron with non-empty interior.*

Proof. It is not hard to see that the set of centerpoints can be obtained as the intersection of closed halfspaces containing $\lceil \frac{3n}{4} \rceil + 1$ points of S and having 3 points of S in its bounding plane. Now, it can be checked that the intersection is not a point (if four such halfspaces had only a point in common, taking complements easily leads to a contradiction) and that it is not contained in a line (this would contradict the result for centerpoints in dimension 2 for the set obtained by projecting perpendicularly to that line). Finally, if the intersection is contained in a plane h, there are $\lceil \frac{3n}{4} \rceil + 1$ points in each of the closed halfspaces bounded by h, and this is only possible if $n \leq 2$. □

Theorem 4.3. *Let S be a set of n points with $h > 4$ extreme points. There exists a set S_1 of n points in simplicial position and such that $E_k(S_1) \leq E_k(S)$ for every $0 \leq k < \frac{n}{4}$.*

Proof. Using Lemma 4.2, we can choose a centerpoint c in the interior of a tetrahedron T whose vertices are extremal points of S. If the convex hull of S has more than four vertices, there exists an extreme point s outside T. Let pqr be the face of T intersected by segment cs. Now, we move points p, q, and r along the rays cp, cq, and cr, respectively. Clearly, in this way the number of extreme points is reduced, so we only have to study how the number of j-facets changes.

We know that the number of j-facets does not change during a convex mutation, so we only have to consider mutations through triangle when we move an extreme point p along the ray cp. Because p is an extreme point, without loss of generality we can consider a mutation $M(u, p, v, w)$ where points c and u are separated by the plane defined by pvw.

If c is contained in the defining side of pvw before the mutation, then the mutation $-M$ has index at least $\frac{n}{4}$ and therefore the vector $E_k(S)$ does not change for $k < \frac{n}{4}$.

On the other hand, if c is not contained in the defining side of pvw and M has index j, according to Lemma 4.1 it follows that if $j = \frac{n-4}{2}$ the vector $E_k(S)$ does not change, while for $j < \frac{n-4}{2}$ we have that $E_j(S)$ decreases by two and $E_k(S)$ does not change for $k \neq j$. □

We are now ready for the main result of this section.

Theorem 4.4. *Let S be a set of n points in $\mathbb{R}^3$ in general position. If $k < \frac{n}{4}$, the number of ($\leq k$)-facets of S is at least $4\binom{k+3}{3}$. Furthermore, this bound is tight.*

Proof. The proof uses induction on n. From Theorem 4.3, we can assume that S is in simplicial position. Let $C = \{p, q, r, s\}$ be the set of vertices of conv(S) and let $S' = S \setminus C$.

Let E_k^j be the number of $(\leq k)$-facets of S adjacent to exactly j vertices of conv(S). Clearly, for $j = 3$, we have four 0-facets (the faces of conv(S)). For $j = 0$, observe that a $(\leq k)$-facet of S' is an $(\leq k+3)$-facet of S. Therefore, using induction we have that that, $E_k^0 \geq 4\binom{k-3+3}{3} = 4\binom{k}{3}$.

For $j = 2$, consider the edge pq. We can rotate the plane defined by pqr around pq and through the set: when the movement has just started, the plane separates r from the rest of points of S. If u_j is the jth point found in this way (and $j \leq \frac{n-3}{2}$), then pqu_j is a j-facet of S. We can repeat the process starting from the plane defined by pqs and, therefore, there are exactly $2k$ $(\leq k)$-facets of S adjacent to pq. Because the j-facets obtained when we repeat this process for the rest of the edges of the convex hull are different, we have that $E_k^2 = 12k$.

Finally, let us consider the case $j = 1$. Let π be the plane passing through qrs and let S'_π be the set obtained by projecting S' from p to π. We observe that if ℓ is a j-edge of S'_π, the plane passing through ℓ and p defines an $(\leq j+2)$-facet of S. Because the number of $(\leq k)$-edges of a set of points in the plane is at least $3\binom{k+2}{2}$ (see [1, 8]), the number of $(\leq k)$-facets of S adjacent to p (and no other extreme point) is at least $3\binom{k}{2}$. Then, $E_k^1 = 12\binom{k}{2}$. Therefore, we obtain

$$E_k(S) = \sum_{i=0}^{3} E_k^i(S) \geq 4 + 12k + 12\binom{k}{2} + 4\binom{k}{3} = 4\binom{k+3}{3}$$

The example in [7] showing that the bound $3\binom{k+2}{2}$ is tight for $k \leq \lfloor \frac{n}{3} \rfloor - 1$ in the planar case can be easily extended to $\mathbb{R}^3$, showing the tightness of the bound. □

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