Binary labelings for bipartite graphs *

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Abstract

Part of the authors introduced in [C. Huemer, S. Kappes, A binary labelling for plane Laman graphs and quadrangulations, in *Proceedings of the 22nd European Workshop on Computational Geometry* 83–86, 2006] a binary labeling for the angles of plane quadrangulations, similar to Schnyder labelings of the angles of plane triangulations since in both cases the labelings are equivalent to tree decompositions. In this paper we analyze an extension of the above labelings to a class of 2-connected bipartite graphs, similar to the extension of Schnyder labelings to 3-connected plane graphs.

1 Introduction

Schnyder labelings are by now a classical tool to deal with planar graphs. A *Schnyder labeling* is a special labeling of the angles of a plane graph with three colors. Schnyder [11] introduced this concept for triangulations, or maximal planar graphs: The angle labeling corresponds directly to a decomposition of the edge set into three spanning trees, or a *Schnyder wood.* Felsner adapted this idea for 3-connected planar graphs [2]. A main application of the Schnyder labeling is a straight-line embedding of a triangulation (on n vertices) on an $n-2$ by $n-2$ grid.

Quadrangulations are maximal bipartite planar graphs and they admit a decomposition of the edge set into two trees. In [6, 7] part of the authors defined a binary labeling for quadrangulations (see Figure 1, middle) that allows to obtain a book embedding for quadrangulations. Tree decompositions and 2-orientations of plane quadrangulations have already received quite some attention; see [5, 8, 9, 10] for work on these topics. The tree decomposition which is implied by the above mentioned binary labeling (see Figure 1, right) has the nice property that at each vertex the two trees are "separated", meaning that from each vertex we can draw two rays which separate the edge sets of the two trees incident to that vertex. Related results about separated trees in quadrangulations have been obtained by de Fraysseix and Ossona de Mendez [4].

2 Strong labelings for quadrangulations

Let G be a plane graph. A *weak labeling* for G is a mapping from the angles of G to $\{0, 1\}$ which satisfies the following conditions:

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(G0) **Special vertices:** There are two special vertices s_0 and s_1 on the outer face of G, such that all angles incident to s_i are labeled i.

(G1) **Vertex rule:** For each vertex $v \notin \{s_0, s_1\}$, the incident labels form a non-empty interval of $1s$ and a non-empty interval of $0s$.

(G2) **Edge rule:** For each edge, the incident labels coincide at one endpoint and differ at the other.

(G3) **Face rule:** For each face (including the outer face), its labels form a non-empty interval of $1s$ and a non-empty interval of $0s$.

A weak labeling of a quadrangulation is a *strong labeling* if it obeys:

(G3$^+$) **Strong face rule:** Each face has exactly one pair of adjacent 0-labels and one pair of adjacent 1-labels. In addition, the edge on the outer face F_{out} which contains s_0 and which has F_{out} to its right when traversed from its white end to the black end has two adjacent labels 0 in F_{out}.

Observation 2.1. A weak labeling induces both a 2-coloring and a 2-orientation of the edges: Every edge is colored according to its endpoint with the two coincident labels and oriented towards that endpoint. Moreover, the vertex rule implies that every vertex except s_0, s_1 has outdegree two; such an orientation will be called *a* 2*-orientation.* See Figure 1 (left). We will identify color 0 with gray and color 1 with black.

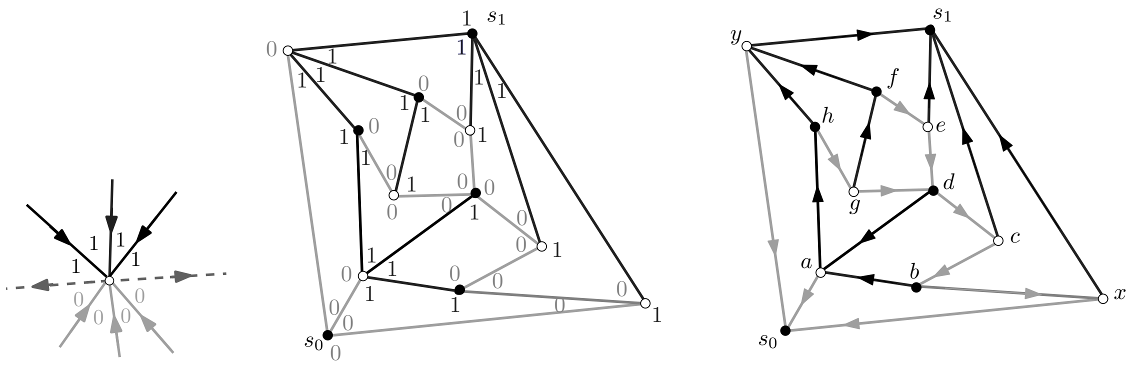

Figure 1: Left: Orientation induced by a weak labeling, the dashed edges may have either color. Middle: Strong labeling of a quadrangulation. Right: Corresponding decomposition into trees.

It follows that a plane graph with n vertices that admits a weak labeling must have exactly $2n-4$ edges. This is the number of edges of a quadrangulation and indeed (see Figure 1, middle):

Theorem 2.2 ([7])**.** *Every quadrangulation admits a strong labeling.*

3 Generalized strong labelings

Weak labelings also exist for some graphs which are not quadrangulations; consider e.g. the graph obtained by inserting into the cycle C_6 the edges 15 and 24. Simple counting shows that further requiring the strong face rule (G3$^+$) implies that G must be a quadrangulation. Hence, in order to have strong labelings on a larger class of graphs we have to modify the rules. The following is inspired by the generalization [2] of Schnyder woods for 3-connected plane graphs.

Definition 3.1. A *strong labeling* for a bipartite plane graph G is a mapping from its angles to the set $\{0,1\}$ which satisfies (G0), (G1), (G2$^+$) and (G3$^+$) for

(G2$^+$) **Extended edge rule:** For each edge, the incident labels form one of the six patterns shown in Figure 2, where black and white vertices are the bipartition classes.

Figure 2: Extended edge rule.

3.1 Merge and split

Bonichon et al. [1] have introduced operations on Schnyder woods which they call *merge* and *split*. A split takes a bidirected edge and opens it up into two unidirected edges. A merge is the inverse operation; it takes an angle with two unidirected edges, one of them incoming the other outgoing, and turns the outgoing edge into the incoming thus making it bidirected. We define similar operations for strong labelings. Figure 3 shows the four possible instances for split and merge. A split is done by replacing a situation from the upper row by the situation below. A merge, conversely, replaces a situation in the lower row by the one above it.

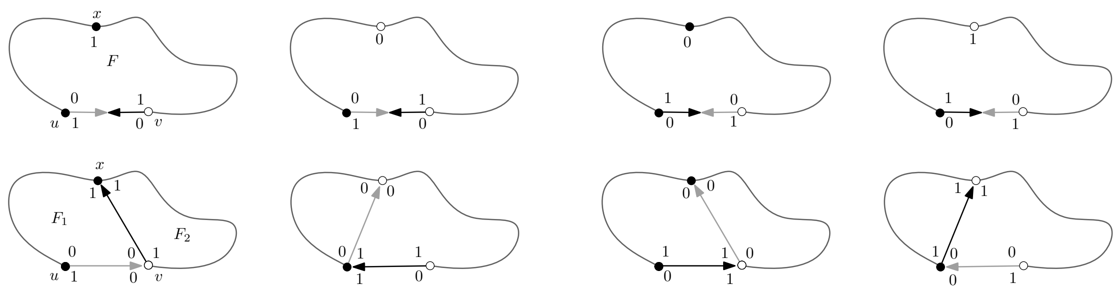

Figure 3: Split and merge for strong labelings.

Lemma 3.2. *If G is a graph with a strong labeling B and a labeling B' of G' is obtained from (G,B) by a split or merge, then B' is a strong labeling of G'.*

Proof. Assuming that B obeys the vertex rules (G0) and (G1) these rules are easily seen to hold for B' as well. All edges in the figure are legal in the sense of (G2$^+$). The least trivial thing is to verify (G3$^+$) for the split. Let us concentrate on the split of the first column where we have given names to the objects. The two black vertices u and x have different labels inside F. Hence, when walking clockwise from the edge vu towards x we have to pass at exactly one of the two edges which have identical labels at both ends inside F, by rule (G3$^+$). Before reaching this edge we always see a 0 at black vertices and a 1 at white vertices. From Figure 2 we find that a clockwise traversal of an edge with identical labels always goes from the white to the black vertex. Hence, the edge we meet has two labels 1. This is what we need to show that (G3$^+$) holds for F_1. Similar arguments show that (G3$^+$) holds for F_2 and indeed that it holds for the two new faces after each of the four possible splits. □

Lemma 3.3. *Let G be a graph with a strong labeling. If G is not a quadrangulation then there is an edge which is feasible for a split.*

Proof. If G is not a quadrangulation then it has more edges than twice the number of faces. Therefore there is a bidirected edge uv. Let u be black and v be white and consider the face F whose clockwise traversal sees e as the edge from v to u. We assume that the label of v in F is 1. From the proof of the previous lemma we deduce that clockwise from vu we reach the edge with labels $1, 1$ and that the second vertex x of this edge is black. This shows that a split of the edge uv towards x is possible. The case in which the label of v in F is 0 works analogously.

A special case occurs if the face F is the outer face. To handle this case think of G as being embedded on the sphere and note that the special conditions of (G3$^+$) for the outer face impose the same structure we have noted for the other faces. Hence splits are possible but special care must be put into the choice of the vertex x towards which an edge is split, a careless choice could split the outer face such that there is no face containing both s_0 and s_1. □

Corollary 3.4. *If G is a graph with a strong labeling then there is a sequence of edge splits which lead to a quadrangulation with a strong labeling.*

Corollary 3.5 (Turning rule)**.** *In a strong labeling of a graph the following is true: If v is a white (respectively black) vertex and uv an incoming edge, then the outgoing edge at v with the same color as uv is the next outgoing edge to the right (respectively left) of uv. See Figure 4.*

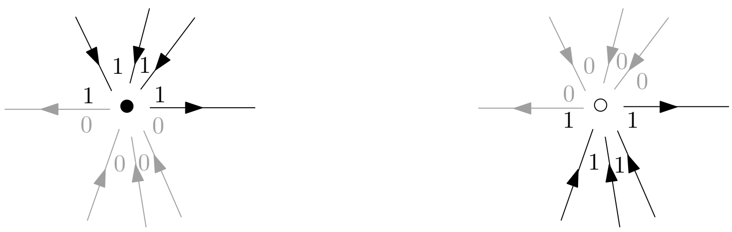

Figure 4: Illustrating the turning rule.

Given a graph G with a strong labeling let T_0 be the set of oriented gray edges and let T_1 be the set of oriented black edges. Let T_i^{-1} be the set of edges of T_i with reversed orientation.

Lemma 3.6. *$T_0 \cup T_1^{-1}$ and $T_1 \cup T_0^{-1}$ are acyclic. Moreover, $T_i, i \in \{0, 1\}$, is a directed tree with sink s_i that spans all vertices but s_{1-i}.*

Proof. Use edge splits to get from G to a quadrangulation Q. The acyclicity of $T_0 \cup T_1^{-1}$ where T_i are the edge sets defined by the orientation of Q was shown in [7]. Note that since a merge has precisely the effect of deleting an edge from $T_0 \cup T_1^{-1}$, this can not introduce cycles. The statement about the trees again follows from the acyclicity of T_i and the fact that every non-special vertex has outdegree one in T_i. □

3.2 Orientations

The orientation induced by a strong labeling on G has the somewhat strange property that it may contain bidirected edges. We encode this orientation by a "regular" orientation of a bigger graph: Let G be a connected bipartite plane graph with distinguished color classes black and white and two special vertices s_0 and s_1 on the outer face. Define a graph S_G as follows: As vertices of S_G take the union of the vertices, edges and faces of G. Every edge-vertex has degree three and is connected to the two endpoints and to the face on its right when traversed from white to black. Figure 5 shows an example. The construction somewhat resembles the primal dual completion of a plane graph as defined in [3].

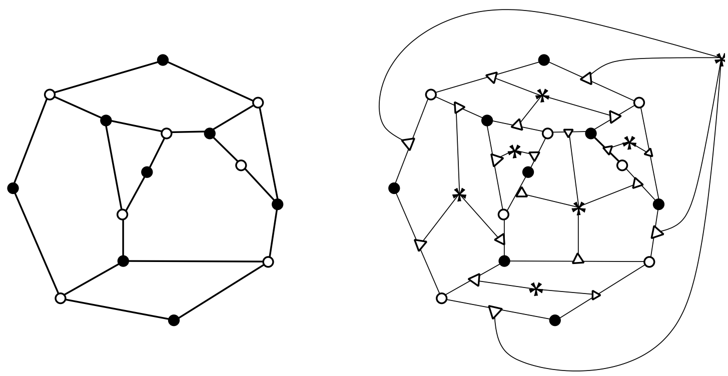

Figure 5: A graph G (left) and the corresponding S_G (right).

Proposition 3.7. *Strong labelings of G are in bijection with orientations of S_G which have the following outdegrees*

$$\mathsf{outdeg}(x) = \begin{cases} 0 & \text{if } x \in \{s_0, s_1\}, \\ 1 & \text{if } x \text{ is an edge-vertex,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Figure 6 shows how to translate from a strong labeling of G to an orientation of S_G. There is a clear correspondence between the rules (G0) and (G1) and the prescribed outdegrees of

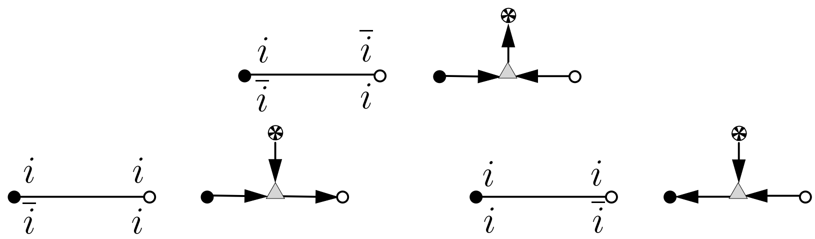

Figure 6: Translating orientations from G to S_G and back.

original vertices. The extended edge rule and outdegree 1 for edge-vertices are both assumed for the translation. The face rule (G3$^+$) corresponds to outdegree 2 for face-vertices. Note that this also holds for the outer-face, the two edges on the outer face which should have repeated labels to confine with (G3$^+$) connect to the vertices s_0 and s_1 which have prescribed outdegree 0. Therefore, these two edge-vertices receive the two outgoing edges of the vertex of the outer face. □

The orientations of S_G described in the proposition are α-orientations in the sense of [3]. Hence, the set of all strong labelings of G can be ordered as a distributive lattice. In particular the strong labelings are flip-connected, where a flip is defined as the complementation of all labels inside a cycle C which is directed in the corresponding orientation of S_G.

3.3 Graphs admitting a strong labeling

So far we have shown that strong labelings have a nice structure. However, we have not yet answered the question of which graphs admit strong labelings. To provide an answer to this question is the goal for this subsection.

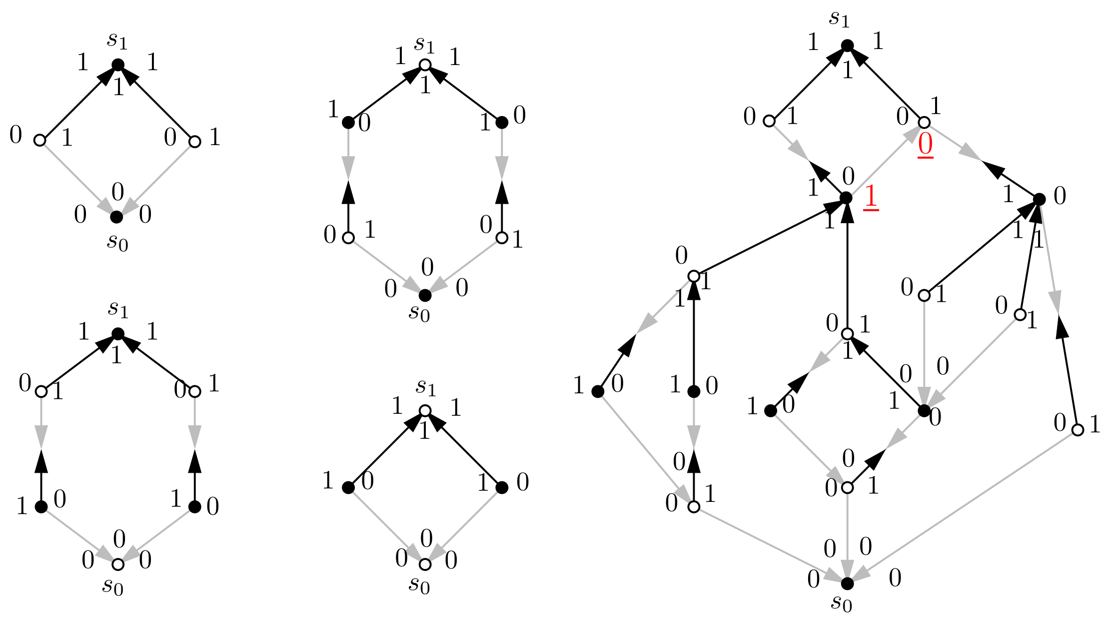

Figure 7: Examples of strong labelings.

To introduce into the topic we have two figures. Figure 7 shows some examples of graphs with strong labelings. The four examples on the left illustrate how the colors of the special vertices influence the labeling along the outer face. The strong labelings in these cases are unique. The strong labeling of the larger graph on the right is not unique, e.g., exchanging the two underlined labels leads to another strong labeling. Figure 8 shows some graphs which do not admit strong

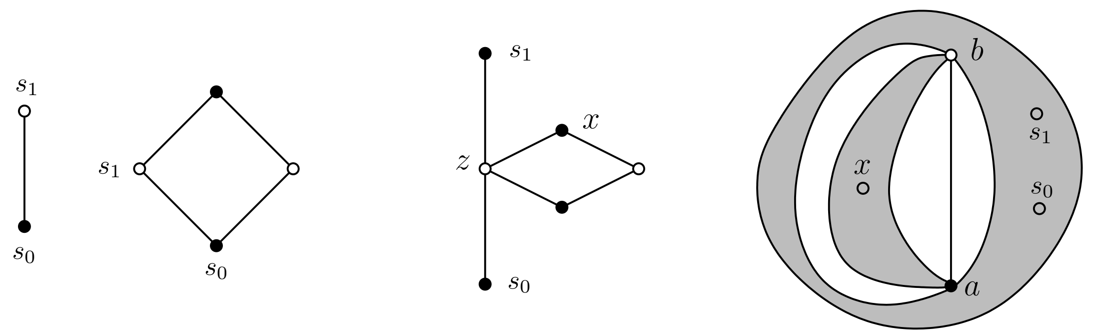

Figure 8: Some graphs which do not admit a strong labeling.

labelings for different reasons. The first two examples fail to admit a strong labeling simply because their two special vertices are adjacent. Rule (G0) would force the connecting edge to have two identical labels on both ends, which is infeasible by the edge rule. In the middle example there is a cut vertex between x and the two special vertices. The two paths $P_0(x)$ and $P_1(x)$ would both contain z which forces a cycle in $T_0 \cup T_1^{-1}$, what is impossible by Lemma 3.6.

An undirected graph with special vertices s_0 and s_1 is called *weakly 2-connected* if it is 2-connected or adding an edge s_0s_1 makes it 2-connected. This is equivalent to saying that every vertex x has a pair of disjoint paths one leading to s_0 and the other to s_1. From the above it follows that being weakly 2-connected is a necessary condition for admitting a strong labeling.

Now consider the sketch on the right of Figure 8. It illustrates the following situation: There is an edge ab, vertex a is black and vertex b white. Removing a and b we disconnect a component C with $x \in C$ from the special vertices s_0 and s_1. Moreover, component C is to the left of ab. If a graph contains such an edge we say that it *contains a block with a right chord*. Suppose that a graph containing a block with a right chord admits a strong labeling. Disjointness forces the two paths $P_0(x)$ and $P_1(x)$ to leave the component C through vertices a and b. From Lemma 3.6 it can be concluded that there is no edge oriented from a or b into C. Now consider the orientation of the edge ab: If it is directed from b to a, then the turning rule for white vertices makes the path $P_i(x)$ leaving at b continue through a where the two paths meet, contradiction. If the direction of ab is from a to b, then it is the turning rule for the black vertex a which leads to the same kind of contradiction.

With the three cases we have identified all the obstructions against admitting a strong labeling:

Theorem 3.8. *Let G be a bipartite plane graph with color classes black and white and two special vertices s_0, s_1 on the outer face. G admits a strong labeling if, and only if, the following conditions are satisfied:*

(1) s_0 and s_1 are nonadjacent,

(2) G is weakly 2-connected,

(3) G contains no block with a right chord.

Proof. The "only if" part comes from the above discussion. The proof for the "if" part is by induction on the number of edges. Let G be a graph satisfying the conditions. We concentrate on the case where s_0 is a black vertex. Let $e = s_0v$ be the edge incident to s_0 which has the duplicate label 0 on the outer face (in our figures it is the leftmost edge at s_0). Remove e from G and let G' be the resulting graph. There are several cases. Figure 9 shows how to deal with them.

The first case is that G' satisfies the conditions and we can by induction assume a strong labeling for G'. Consider the edge uv on the boundary of the outer face of G' which is interior in G. In the labeling of G' on the outer face the black vertex u has label 1 and the white vertex v has label 0. The extended edge rule (G2$^+$) implies that the labels on the opposite side of this edge are inverse, 0 at u and 1 at v. Therefore, it is consistent with edge and vertex rules to label the angle between e and uv with 1 and the outer angle of e at v with 0. This yields a strong labeling of G.

If G' does not satisfy the conditions then, necessarily, it is condition (2) which fails. If G' is not connected it has s_0 as an isolated vertex. Choose v as the special vertex s_0' for the component of G' which contains s_1. If this component admits a strong labeling we can extend this to the full graph. Otherwise, condition (1) is not satisfied. Hence either the component is just the single edge $s_0's_1$ or this edge is a left chord to a block which satisfies all three conditions. In both cases it is easy to get to a strong labeling of G.

If G' is connected but fails to satisfy (2), then it has a cut vertex. Let w be the cut vertex such that one of the components is weakly 2-connected between v and w and the other is weakly 2-connected between s_0 and s_1. The first of these components is either a single edge or it satisfies the conditions. The second component also satisfies the conditions. By induction both components have strong labelings. Again it is straightforward to define a strong labeling based on the strong labelings of the components. The right part of Figure 9 shows the case where w is white. □

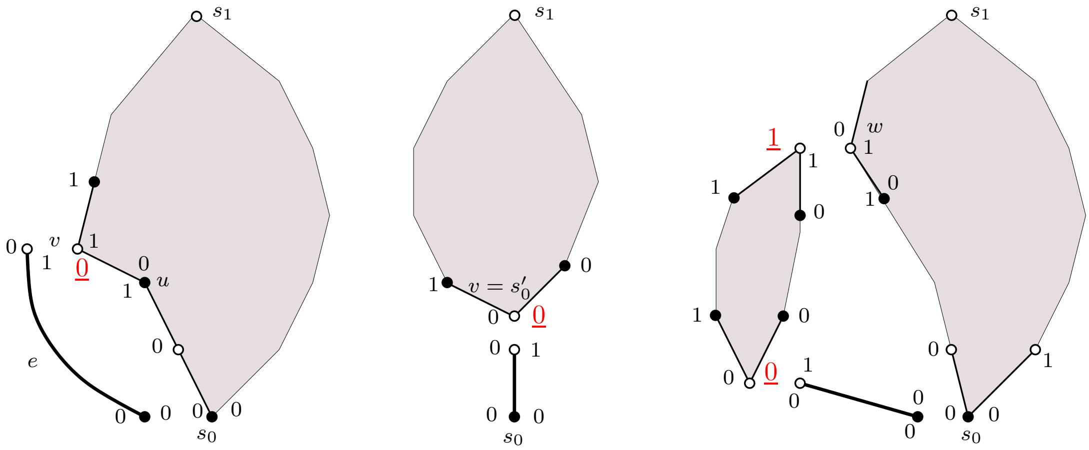

Figure 9: Constructing the strong labeling in the inductive proof. Underlined labels are inverted in the labeling of G.

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