Minimum Illumination Range Voronoi Diagrams*

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Abstract

The MIR Voronoi diagram appears with the notion of *good illumination* introduced in [3, 4]. This illumination concept generalizes *well-covering* [7] and triangle guarding [9]. The MIR Voronoi diagram merges the notions of proximity and convex dependency. Given a set S of planar light sources, for each point q in the plane we search for the subset $S_q \subset S$ *nearest* to q and such that q is an interior point of the convex hull of S_q. The furthest point to q in S_q is called the Minimum Illumination Range point (MIR point) of q with respect to S. The MIR Voronoi diagram splits the interior of the convex hull of S into several regions, associating each point in S to its MIR point. This diagram is motivated by optimization problems on good illumination.

Key words: Voronoi Diagrams, Good Illumination, Minimum Illumination Range.

1. Introduction

Illumination, also known as guarding or covering, is a well studied subject in Computational Geometry (see [10] for a survey). The basic idea is the following: one point-light p illuminates another point q whenever the line segment with endpoints p and q is unobstructed. Many generalizations of this concept have been studied. Some of them consider different kinds of obstacles and others consider different kinds of visibility instead of the usual straight line visibility. There are also papers considering quality conditions for the illumination of objects. Canales et al. [3, 4] define good illumination, a generalization of both triangle guarding [9] and well-covering [7].

Definition 1. *Let S be a set of n light sources in the plane. We say that a point q in the plane is t-well illuminated by S if, and only if, every half-plane containing q in its interior, contains at least t points of S illuminating q.*

The motivation behind this definition is the fact that, in some applications, it is not sufficient to have one point illuminated. It is necessary to have several light sources illuminating some neighbourhood of the point [7]. This way, the greater the number of light sources in every half-plane containing the point q, the better the illumination of q.

Canales [4] studies several problems related to good illumination involving light sources and obstacles in the plane. In real life applications, light sources do not have unlimited illumination range, so we add another quality condition (see [1] or [2]) to the light sources: the limited illumination range condition. These two papers consider optimization problems related with limited illumination range light sources. The main goal is to compute the Minimum Illumination Range (MIR) that light sources must have in order to well illuminate a set of points. Let us consider the following problem. Given a set $S = \{s_1, \dots, s_n\}$ of fixed light sources in the plane and a planar point q, we want to compute which light sources of S we are supposed to turn on to have q 1-well illuminated with the minimum light source power. These light sources have the same illumination range. The solution to this problem is given in [2] with an $\mathcal{O}(n \log n)$ algorithm that computes a Minimum Illumination Range triangle that 1-well illuminates q. A MIR triangle is a triangle containing q in its interior and that minimizes the

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maximum distance between q and the furthest triangle's vertex to q.

If the planar point q moves continuously or if it changes quickly from one spot to another, we have to recompute the correspondent MIR triangle as many times as necessary to maintain point q 1-well illuminated. This problem generated the following question: how do we preprocess the point set S to get a quick solution?

For a fixed position of point q, the Minimum Illumination Range needed to 1-well illuminate q is determined by the nearest point to q, $s_q \in S$ that allows q to be an interior point of the convex hull of the subset $S_q \subset S$. This set contains s_q and all the points in S that are closer to q than s_q. The point s_q is called the MIR point of q with respect to S. The MIR point of q determines the Minimum Illumination Range needed to 1-well illuminate q in each position. This condition motivates the following definition of the MIR Voronoi diagram.

Definition 2. *Let $S = \{s_1, \ldots, s_n\}$ be a set of light sources in the plane. For every light source $s_i \in S$, the MIR-Voronoi region of s_i with respect to the set S is the set MIR-$VR(s_i, S) = \{x \in \mathbb{R}^2 :$ MIR point of x with respect to S is $s_i\}$. The set of all the MIR Voronoi regions is called the MIR Voronoi diagram of S and will be denoted by $MIRVD(S)$.*

To have a lighter notation, from now on $MIR-VR(s, S)$ will be denoted as $VR(s, S), s \in S$. In Figure 1 there is an example of the diagram with four light sources. In this case, the 1-well illuminated area is a quadrangle. An obvious observation is that Voronoi regions are strictly contained in the convex hull of S. The MIR Voronoi regions are not necessarily convex (see Figure 1) and the edges between them are line segments.

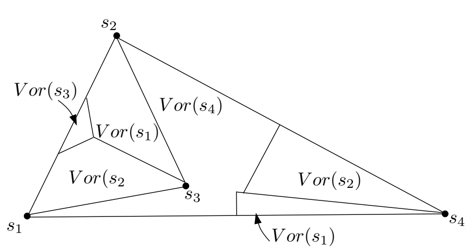

Figure 1. MIR Voronoi diagram of a set of four points. Note that $s_3 \in CH(S)$.

The MIR Voronoi diagram merges two very different geometric concepts: convex dependency and proximity. Every interior point of the convex hull of S is associated with the nearest point in S that verifies the convex dependency condition. Many interesting questions that arise from this new concept are ongoing work. The MIR Voronoi diagram is a structure that allows us to compute the Minimum Illumination Range for a point in $\mathcal{O}(\log n)$ time just by locating the point in the diagram. In section 2 we give some properties of the MIR Voronoi diagram in order to better understand the concept. In section 3 we give two different algorithms for computing the diagram. We conclude this paper with the conclusions and future work section, followed by the acknowledgments.

2. Properties of the MIR Voronoi diagram

In this section we present some properties of the MIR Voronoi diagram. Due to their simplicity, some proofs will be omitted. The first lemma relates 1-good illumination and MIR triangles.

Lemma 1. *Let S be a set of n light sources in the plane.*

1. *All the interior points of the convex hull of S are 1-well illuminated.*
2. *Let q be an interior point of the convex hull of S. Except for the degenerate case described below, q is 1-well illuminated by S if, and only if, there is a MIR triangle in S that 1-well illuminates q.*

The mentioned degenerate case is the following: point q is such that the four nearest points of S to q are in convex position and the two diagonals of the convex hull of these four points intersect exactly at point q. Note that there is not a MIR triangle for a point in these conditions. We exclude this possibility for point q. Nevertheless, considering MIR quadrilaterals instead of MIR triangles, all the results are valid for this degenerated case.

Proposition 1. *The light sources are vertices of the MIR Voronoi diagram.*

The following lemma has as a consequence that interior lights are vertices of the diagram.

Lemma 2. *If s is an interior light source of the $CH(S)$ then s is a reflex vertex of a Voronoi region.*

Proof: Suppose that $s \in VR(s_f, S)$, this is, $s_f \in S$ is the furthest vertex of a MIR triangle that 1-well illuminates s. Let $S' \subset S$ be the subset of light sources of $S \backslash \{s\}$ contained in D, where D is the disk of radius $d(s, s_f)$ centered at s (see Figure 2). By the $VR(s_f, S)$ definition, $s \notin CH(S')$. Let s' and s'' in S be the support points of s for $CH(S')$. Let $s^* \in S$ be the nearest light source to s outside of the disk D and let $\epsilon = \frac{1}{2}(d(s, s^*) - d(s, s_f))$. Consider a disk $D' \subset T(s', s'', s_f)$ centered at s whose radius is smaller than ϵ. Each point $z \in D'$ situated in the convex sector determined by the points s', s and s'' does not belong to $VR(s_f, S)$ as $T(s, s', s'')$ is the MIR-Triangle that 1-well illuminates z and s_f is not one of its vertices. Each point $x \in D'$ situated in the reflex sector determined by the points s', s and s'' belongs to $VR(s_f, S)$ as $T(s_f, s', s'')$ is a MIR triangle to x. Therefore s is a reflex vertex of $VR(s_f, S)$. □

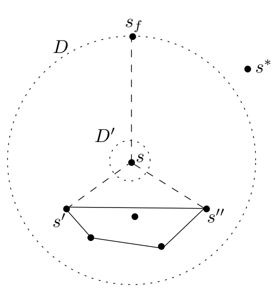

Figure 2. Light source $s \in S$ is the vertex of some regions in the MIRVD.

Note that a light source never is a vertex of its Voronoi region. If a light source were a vertex of its own Voronoi region then there would be a MIR triangle that 1-well illuminates it and the furthest vertex of that triangle would be itself. This is impossible since the other two vertices of the triangle must be further to the point than the point itself.

Proposition 2. *: The Voronoi region of a light source $s \in S$ is radially monotone (i.e. the intersection between $VR(s, S)$ and any half-line with origin in s is a line segment or an empty set).*

Proof: For every point q in the plane, let q_f be the furthest vertex of a MIR triangle that 1-well illuminates q, that is the site whose MIR-Voronoi region contains q, and C_q the circle centered at q that contains q_f on its boundary. One verifies that C_q contains a diameter separating q_f from all the light sources lying in the interior of C_q and there is, at least, one light source interior to C_q in each half circle determined by the line passing through q and q_f (see Figure 3(a)).

Let a and b be two different points on a half-line with endpoint s such that $d(s, a) < d(s, b)$ and that a and b are in $VR(s, S)$, that is, $a_f = b_f = s$. This implies that C_a is contained in C_b and both circles are tangent to point s. Furthermore, there must be two other light sources v and w in the interior of C_a such that v, w and s are the vertices of a MIR triangle for point a. Light sources v and w cannot be in the empty half circle of C_b and, as they are on different half planes with respect to $\overline{sa}$ (this line also goes through b since s, a and b are colinear), v, w and s also determine a MIR triangle for b and for every point x lying on the line segment $\overline{ab}$ (see Figure 3(b)). □

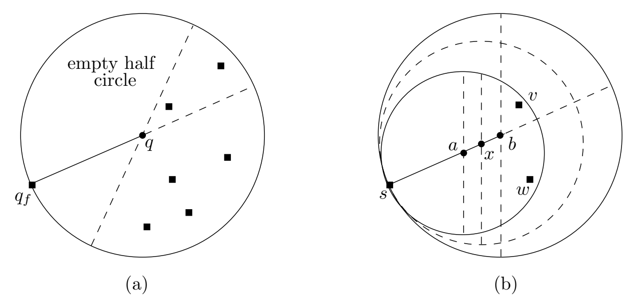

Figure 3. (a) C_q is a circle centered at q that contains the MIR point of q, q_f, on its boundary. (b) The triangle with vertices v, w and s is a MIR triangle for all the points in $\overline{ab}$.

Let S be a set of n light sources where $(cos(\frac{\pi}{2^{i-1}}), sin(\frac{\pi}{2^{i-1}}))$ are the coordinates of s_i. This means that the light sources of S are located on the circumference of radius 1 centered at $(0, 0)$.

Lemma 3. *With the above configuration of lights, the light sources $s_1, s_2, \ldots, s_{k-1}$ are not necessary to compute the MIRVD restricted to the triangle $T(s_k, s_{k+1}, s_n)$, for all $k = 2, \ldots, n-2$.*

For each triangle $T(s_k, s_{k+1}, s_n)$ consider the set of triangles T_{kj}, $j = k+1, \ldots, n$, determined by the line segments $\overline{s_k s_j}$, $\overline{s_k s_{j+1}}$ and the perpendicular bisector between s_k and s_{j+1}. The triangles

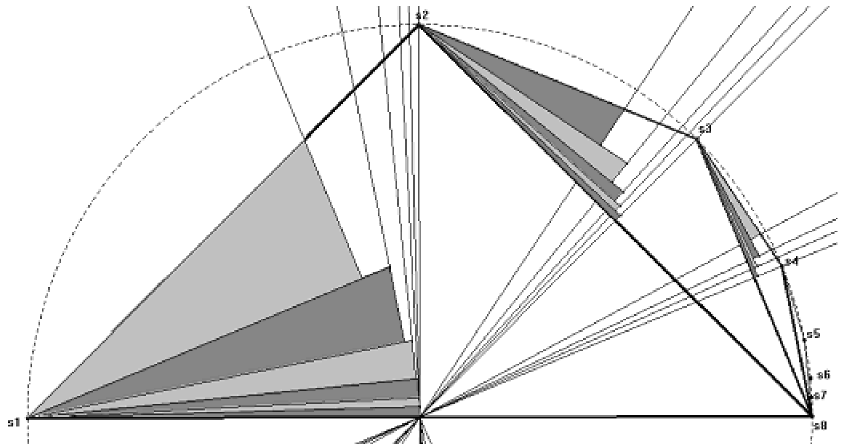

Figure 4. S and the set of fans whose extreme vertices are s_1, s_2 and s_3.

T_{kj} are fans whose extreme vertex is s_k (see Figure 4).

Lemma 4.

1. *The triangle* $T_{kj} \subset VR(s_{j+1}, S)$, *for all* $k = 1, \ldots, n-3$ *and for all* $j = k+1, \ldots, n-1$.
2. *The points of the triangle* $T(s_k, s_{k+1}, s_n)$ *that are not in the set of fans* T_{km} *of the vertex* s_k *belong to* $VR(s_k, S)$, *this is,*

$$T(s_k, s_{k+1}, s_n) - \bigcup_{m=k+1}^{n} T_{km} \subset VR(s_k, S).$$

Proposition 3. *The MIR Voronoi diagram can have quadratic complexity.*

Proof: The set S described as above has a linear number of regions of size $\mathcal{O}(n)$ and a linear number of regions with a linear number of connected components. □

Proposition 4. *Given the MIR Voronoi diagram of a set* S,

1. *the MIR point of any point in the plane with respect to* S *can be computed in* $\mathcal{O}(\log\ n)$ *time*
2. *a MIR triangle of any interior point of the convex hull of* S *can be computed in* $\mathcal{O}(n)$ *time.*

Proof:

1. It suffices to locate the point in the MIR Voronoi diagram, which can be done in $\mathcal{O}(\log\ m)$ time being m the size of the planar partition. As the MIR Voronoi diagram has a polynomial size with respect to the number of light sources, $\mathcal{O}(\log\ m) = \mathcal{O}(\log\ n)$. □

2. We compute the furthest vertex s_f of a MIR triangle that 1-well illuminates s in $\mathcal{O}(\log\ n)$ time. After that, we can compute in linear time two light sources that are inside the circle centered at s with the distance between s and s_f as its radius and such that each light source is in a different half circle with respect to $\overline{ss_f}$. Every pair of light sources in these conditions together with s_f define a MIR triangle for point s and this whole process requires $\mathcal{O}(n)$ time. □

3. Algorithms

We follow by presenting two algorithms to compute the MIR Voronoi diagram. The first algorithm divides the plane in a similar way to the one we did on the algorithm *MIR-Segment* [1, 2] and runs in $\mathcal{O}(n^5\ \log\ n)$ time. The second algorithm is faster and takes advantage of the ordinary Voronoi Diagrams. It requires $\mathcal{O}(n^5)$ time.

3.1. The First Algorithm

We now present the first algorithm to compute the MIR Voronoi diagram (MIRVD). This algorithm divides de plane into several regions. Each region is assigned the same colour as the furthest vertex in a MIR triangle that 1-well illuminates it. There is an example of this algorithm in Figure 5.

Let S be a set of n light sources in the plane. There is an auxiliary algorithm called *MIR-Point*

[1, 2] that runs in $\mathcal{O}(n \log n)$ time as explained before.

Input: *A set S of n light sources in the plane.*
Output: *MIRVD, the MIR Voronoi diagram.*

1. Assign each light source with a different colour.
2. Compute the arrangement of the line segments connecting every pair of light sources and of all the lights' perpendicular bisectors. This procedure divides the plane in several regions.
3. **For** each region within the $CH(S)$ **do**

   Choose an interior point of the region and compute the algorithm *MIR-Point* to find the furthest vertex in the MIR triangle.

   Assign all the points in the region to the colour of the light source just found.

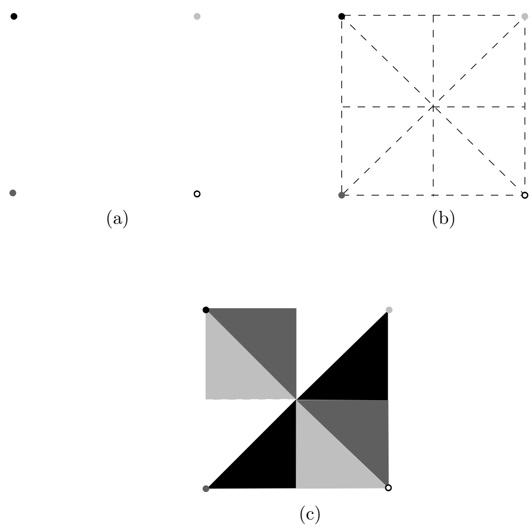

Figure 5. (a) Each light source has a different colour. (b) The plane is divided in eight regions by the perpendicular bisectors and the line segments connecting all the light sources. (c) The final MIRVD for the points inside the $CH(S)$.

Proposition 5. *Given a set S of n light sources in the plane, the algorithm described above computes the $MIRVD$ in $\mathcal{O}(n^5 \log n)$ time.*

Proof:

Correctness: Suppose that we have a coloured region R and that there is, at least, one point $r \in R$ whose furthest vertex in some MIR triangle that 1-well illuminates r has a different colour. This means that r is not in the same MIR triangle as some of its neighbouring points or that r is in the same MIR triangle as its neighbours but the furthest vertex is not the same. r must be inside the same MIR triangle as its neighbours since they are all in R. This could only happen if r and its neighbours were in opposite sides of a line segment connecting two light sources, this is, r would not be in the same region as its neighbours. Then r must have a different furthest vertex from its neighbours in the same MIR triangle. This means that the perpendicular bisector between the furthest vertex to r and the furthest vertex to its neighbours must separate the r from its neighbours, this is, they must be in different regions.

We conclude that every point in the same computed region has the same furthest vertex in a MIR triangle that 1-well illuminates them, this is, every point in the same region has the same colour.

Complexity Analysis: Step 1 can be carried out in $\mathcal{O}(n)$. Step 2 takes $\mathcal{O}(n^2)$. Step 3 requires $\mathcal{O}(n^5 \log n)$ since we have to compute the arrangement of a quadratic number of lines which generate up to $\mathcal{O}(n^4)$ regions. Applying the algorithm *MIR-Point* to each region gives us the final complexity. □

3.2. The Second Algorithm

Finally, we present the best algorithm known so far to compute the MIR Voronoi diagram (MIRVD). This algorithm takes advantage of the ordinary Voronoi Diagrams. Starting with the order 3 Voronoi Diagram, we know that each point inside the same region has the same three nearest light sources, this is, each point is illuminated by three light sources. To have these points 1-well illuminated we need to check if they are inside the convex hull of the three nearest light sources to them. We keep all the points that are 1-well illuminated. After we are done with the order 3 Voronoi Diagram, we get some possibly disconnected regions where the points are 1-well illuminated by the three nearest light sources to them. Then we need to find the furthest light source to each point, this is, we need to compute the Furthest Voronoi Diagram to each region. After this, we are left with the points that are not 1-well illuminated by the three nearest light sources. Then we check whether the four nearest light sources are sufficient to 1-well illuminate the points left and repeat the described procedure. If four light sources are not enough, we continue increasing the number of light sources until, in the worst case, we reach the Furthest Voronoi Diagram. It is easy to find an example where the furthest light source in the MIR triangle that 1-well illuminates a point is indeed the furthest of them all. We stop when we have all the possible points coloured. There is an example of this algorithm in Figure 6.

Input: *A set S of n light sources in the plane.*
Output: *MIRVD, the MIR Voronoi Diagram.*

1. Compute all the order k Voronoi Diagrams, $3 \leq k \leq n-1$.
2. $D \leftarrow \emptyset$, $k \leftarrow 3$.
3. **While** $Area(D) < Area(CH(S))$ **do**
   For each region R of the k order Voronoi Diagram **do**
   $R \leftarrow R \backslash D$.
   Compute the intersection I between the region R and the convex hull of the k nearest light sources to it.
   If $I \neq \emptyset$ **do**
   Compute the Furthest Voronoi Diagram of I.
   $D \leftarrow D \cup I$, $k \leftarrow k+1$.
4. $MIRVD \leftarrow D$.

Proposition 6. *Given a set S of n light sources in the plane, the algorithm described above computes the $MIRVD$ in $\mathcal{O}(n^5)$ time.*

Proof:
Correctness: To prove that this algorithm also works, suppose that we have a region R and that there is, at least, one point $r \in R$ whose furthest vertex in a MIR triangle that 1-well illuminates r is different from the furthest vertex on a MIR triangle that 1-well illuminates its neighbours. The point r is in the intersection between an order k diagram and the convex hull of its k nearest lights. As we are not working with the regions already 1-well illuminated, r is not 1-well illuminated by its $k-1$ nearest light sources, this is, r is not inside the convex hull of its $k-1$ nearest light sources. Following this, we know that the k^{th} nearest light source to r is the one that makes it possible for r to be 1-well illuminated, this is, the k^{th} nearest light source to r is the furthest vertex of a MIR triangle that 1-well illuminates it. If we compute the Furthest Voronoi Diagram and the furthest vertex on a MIR triangle that 1-well illuminates r is different from the furthest vertex on a MIR triangle that 1-well illuminates its neighbours, then r does not belong to R.

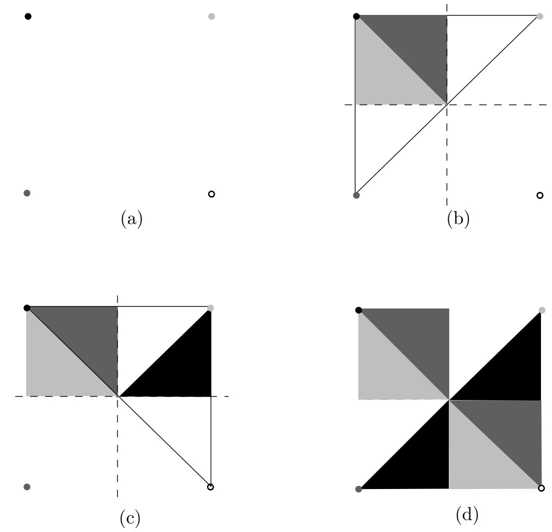

Figure 6. (a) Each light source has a different colour. (b) The 3 order Voronoi Diagram is represented by the dashed lines. The first intersection between a triangle formed by three light sources and one of the regions of the 3 order Voronoi Diagram after the colouring. (c) The second intersection between a triangle formed by three light sources and one of the regions of the 3 order Voronoi Diagram after the colouring. (d) The final MIRVD.

We conclude that every point in the same computed region has the same furthest vertex in a MIR triangle that 1-well illuminates them.

Complexity Analysis: Step 1 can be carried out in $\mathcal{O}(n^3)$ [6]. Each Voronoi Diagram can have up to $\mathcal{O}(n^2)$ regions. Though the Voronoi regions are convex, the clipping might result on a non convex-region (polygon with a hole). This way, the intersection of the two regions requires $\mathcal{O}(n^2)$ and the Furthest Voronoi Diagram can be computed in $\mathcal{O}(n \log n)$. Following this, step 2 requires $\mathcal{O}(n^4)$ for each Voronoi Diagram. We get to an overall complexity of $\mathcal{O}(n^5)$. Step 3 has a constant complexity. □

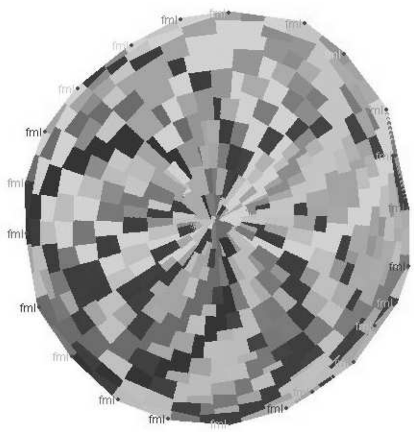

Figure 7. An example of a MIRVD when the set S is in convex position.

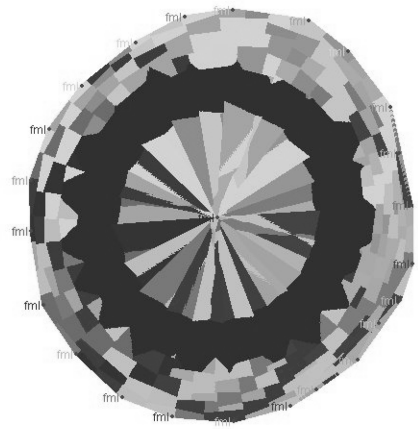

Figure 8. The Voronoi region of the point in the middle has a hole.

4. Examples

In this section we present some examples of the MIRVD. The originals are coloured but these are still fascinating. In Figure 4 we can see an example of a MIRVD with a quadratic number of regions. In Figure 7 we can see an example of a MIRVD when the set S is in convex position and its points are in a degenerate circular position. In Figure 8 we have the same set of points in convex position but we added one point in the middle. It is quite clear that the Voronoi region of this new point has a hole. Actually, these Voronoi regions cannot have more than one hole because of their radial monotonicity.

5. Conclusions and Further Work

Let us face the problem that motivated the MIRVD structure. The problem we wanted to efficiently solve was to compute a set of light sources among the n given that we need to turn on in each moment in order to maintain a point q well illuminated. The point q is continuously changing its position or jumping to another spot in the plane. As we can select the light sources' illumination range, we want to optimize the power needed to illuminate q. We consider the case where all the light sources have the same power and we also consider that the light sources' power is linearly dependent to the distance between a light source and the furthest point it illuminates.

The idea is to turn our problem into a discrete one by keeping the position of the moving point at discrete instant times. It is possible to solve the problem for a fixed position in $\mathcal{O}(n \log n)$ time by computing a MIR triangle for q with respect to the set of light sources (see [2]). This rises a $\mathcal{O}(kn \log n)$ algorithm where k is the number of points after the movement of q has been discretized.

If the MIR Voronoi diagram is computed in a prior step, we can locate a point q in the diagram and know its MIR point in $\mathcal{O}(\log n)$ time. As we have seen, a MIR triangle can be computed in $\mathcal{O}(n)$ time. So, by using the MIRVD the total amount of time needed for k discrete steps is $\mathcal{O}(kn)$.

Note that if all the light sources are turned on permanently and we can only vary the power of all of them at the same time, the Minimum Illumination Range needed in each moment can be computed in $\mathcal{O}(\log n)$ time. We can lower this complexity and make it constant for every step (except for the first one) in the practical cases where the point q moves continuously following a known trajectory. Note that if a point lies in a region of the MIRVD, it can only move to its adjacent regions.

We are working to obtain more properties of the MIR Voronoi diagrams as well as on improving the algorithms' complexities. Lower bounds are also being studied. We know that the MIR point problem is not a LP type problem, which suggests that the lower bound will fit the $\mathcal{O}(n \log n)$ upperbound we provide. We also conjecture that the $\Omega(n^2)$ lower bound we have for the size of the

MIRVD is tight.

This is an ongoing work which is part of [8].

6. Acknowledgments

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