Chapter 1

Formal Pushdown Automata

Formal Definition and View

Lecture *Formal Pushdown Automata* on the 28th April 2009

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Aim of the Lecture

1. Define pushdown automaton in a formal way
2. Top-down and bottom-up analysis of context-free languages
3. One-turn pushdown automata
4. Atomic pushdown automata

Contents

What is a *pushdown*?

Merriam-Webster

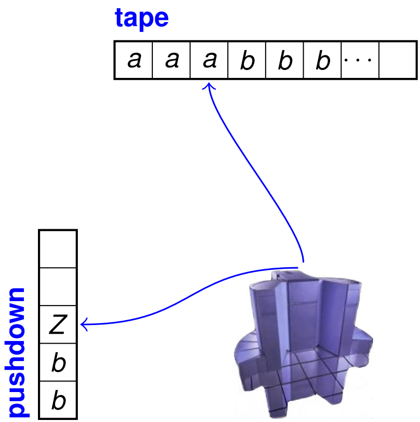
A store of data (as in a computer) from which the most recently stored item must be the first retrieved—called also *pushdown list* or *pushdown stack*.

We stick to this definition!

Why do we need them?

- Finite automata handle regular languages only
- Syntax of programming languages is context-free
  - Program text is of arbitrary length
  - Repeated nested bracket-like structures (e.g. arithmetic expressions)
- Pushdown automata offer infinite memory (even if LIFO structure)

Informal View



Definition of Pushdown Automaton

Formal Definition of Pushdown Automaton

A pushdown automaton is a seven-tuple
$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where

- Q — finite *set of states*
- Σ — finite *input alphabet*
- Γ — finite *alphabet of pushdown symbols*
- δ — mapping $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ *transition function*
- $q_0 \in Q$ — *starting/initial state*
- $Z_0 \in \Gamma$ — *start symbol* on the pushdown
- $F \subseteq Q$ — *set of final states*

Configuration

Definition of Configuration

A *configuration* of PA M is a triple $(q, w, \alpha) \in Q \times \Sigma^* \times \Gamma^*$, where

- q — current state of the finite control
- w — unused input; the first symbol of w is under the reading head; if $w = \varepsilon$, then it is assumed that all the input has been read
- α — contents of the pushdown; the leftmost symbol is the topmost symbol; if $\alpha = \varepsilon$, then the pushdown is assumed to be empty

Move / Transition

Move/Transition Definition

A *move/transition* by M is represented by the binary relation $\vdash_M$ (or $\vdash$ if M is understood) on configurations. We write

$$(q, aw, Z\alpha) \vdash (q', w, \gamma\alpha)$$

if $(q', \gamma) \in \delta(q, a, Z)$ for any $q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Sigma^*$, $Z \in \Gamma$, and $\alpha \in \Gamma^*$.

We can define relations $\vdash^i$, $i \geq 0$, $\vdash^*$, and $\vdash^+$ in the traditional way. $\vdash^*$ and $\vdash^+$ stand for reflexive-transitive and transitive closures of $\vdash$, respectively.

Final Configuration

- Initial configuration: (q_0, w, Z_0), $w \in \Sigma^*$
- *Note: from the definition, it follows that there is no move possible if the pushdown is empty.*
- Final configuration, Ξ, $\Xi \in \{1, 2, 3\}$:
  1. Empty pushdown: $(q, \varepsilon, \varepsilon)$, $q \in Q$
  2. Final state: $(q_F, \varepsilon, \alpha)$, $q_F \in F$, $\alpha \in \Gamma^*$
  3. Final state and empty pushdown: $(q_F, \varepsilon, \varepsilon)$, $q_F \in F$

Accepted Language

- A word w, $w \in \Sigma^*$, is accepted by M if it makes transitions $(q_0, w, Z_0) \vdash^* \Xi$, where $\Xi \in \{1, 2, 3\}$ is the selected final configuration.

Language accepted by *M*

$L(M) = \{w \mid (q_0, w, Z_0) \vdash^* \Xi\}$, where $\Xi \in \{1, 2, 3\}$ is some (but fixed) of the final configurations.

Example 1 ($a^n b^n$, $n \geq 0$) — Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0\}$
$\Sigma = \{a, b\}$
$\Gamma = \{a, Z_0\}$
$F = \{q_0\}$

δ: $\delta(q_0, a, Z_0) \to \{(q_0, Z_0 a)\}$
$\delta(q_0, b, a) \to \{(q_0, \varepsilon)\}$
$\delta(q_0, \varepsilon, Z_0) \to \{(q_0, \varepsilon)\}$

Example 1 ($a^n b^n$, $n \geq 0$) — Acceptance

$(q_0, aaabbb, Z_0)$	// $\delta(q_0, a, Z_0) \rightarrow (q_0, Z_0 a)$
$\vdash (q_0, aabbb, Z_0 a)$	// $\delta(q_0, a, Z_0) \rightarrow (q_0, Z_0 a)$
$\vdash (q_0, abbb, Z_0 aa)$	// $\delta(q_0, a, Z_0) \rightarrow (q_0, Z_0 a)$
$\vdash (q_0, bbb, Z_0 aaa)$	// $\delta(q_0, \varepsilon, Z_0) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, bbb, aaa)$	// $\delta(q_0, b, a) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, bb, aa)$	// $\delta(q_0, b, a) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, b, a)$	// $\delta(q_0, b, a) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, \varepsilon, \varepsilon)$	

Example 1 ($a^n b^n$, $n \geq 0$) — Failure

$$
\begin{array}{ll}
(q_0, aabbb, Z_0) & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, Z_0 a) \\
\quad \vdash (q_0, abbb, Z_0 a) & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, Z_0 a) \\
\quad \vdash (q_0, bbb, Z_0 aa) & // \ \delta(q_0, \varepsilon, Z_0) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, bbb, aa) & // \ \delta(q_0, b, a) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, bb, a) & // \ \delta(q_0, b, a) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, b, \varepsilon) & // \text{ no rule to be applied}
\end{array}
$$

Example 2 ($ww^R, w \in \{a,b\}^+$) — Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0, q_1, q_F\}$
$\Sigma = \{a, b\}$
$\Gamma = \{a, b, Z_0\}$
$F = \{q_F\}$

$$\begin{aligned}
\delta:\ & \delta(q_0, a, Z_0) \rightarrow \{(q_0, aZ_0)\} \\
& \delta(q_0, b, Z_0) \rightarrow \{(q_0, bZ_0)\} \\
& \delta(q_0, a, a) \rightarrow \{(q_0, aa), (q_1, \varepsilon)\} \\
& \delta(q_0, a, b) \rightarrow \{(q_0, ab)\} \\
& \delta(q_0, b, a) \rightarrow \{(q_0, ba)\} \\
& \delta(q_0, b, b) \rightarrow \{(q_0, bb), (q_1, \varepsilon)\} \\
& \delta(q_1, b, b) \rightarrow \{(q_1, \varepsilon)\} \\
& \delta(q_1, a, a) \rightarrow \{(q_1, \varepsilon)\} \\
& \delta(q_1, \varepsilon, Z_0) \rightarrow \{(q_F, \varepsilon)\}
\end{aligned}$$

Example 2 ($ww^R, w \in \{a,b\}^+$) — Acceptance

$(q_0, abbaabba, Z_0)$	$// \delta(q_0, a, Z_0) \to (q_0, aZ_0)$
$\vdash (q_0, bbaabba, aZ_0)$	$// \delta(q_0, b, a) \to (q_0, ba)$
$\vdash (q_0, baabba, baZ_0)$	$// \delta(q_0, b, b) \to (q_0, bb)$
$\vdash (q_0, aabba, bbaZ_0)$	$// \delta(q_0, a, b) \to (q_0, ab)$
$\vdash (q_0, abba, abbaZ_0)$	$// \delta(q_0, a, a) \to (q_1, \varepsilon)$
$\vdash (q_1, bba, bbaZ_0)$	$// \delta(q_1, b, b) \to (q_1, \varepsilon)$
$\vdash (q_1, ba, baZ_0)$	$// \delta(q_1, b, b) \to (q_1, \varepsilon)$
$\vdash (q_1, a, aZ_0)$	$// \delta(q_1, a, a) \to (q_1, \varepsilon)$
$\vdash (q_1, \varepsilon, Z_0)$	$// \delta(q_1, \varepsilon, Z_0) \to (q_F, \varepsilon)$
$\vdash (q_F, \varepsilon, \varepsilon)$	

Example 2 ($ww^R, w \in \{a, b\}^+$) — Failure

$$
\begin{aligned}
(q_0, abbaaaba, Z_0) & & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0) \\
& \vdash (q_0, bbaaaba, aZ_0) & // \ \delta(q_0, b, a) \rightarrow (q_0, ba) \\
& \vdash (q_0, baaaba, baZ_0) & // \ \delta(q_0, b, b) \rightarrow (q_0, bb) \\
& \vdash (q_0, aaaba, bbaZ_0) & // \ \delta(q_0, a, b) \rightarrow (q_0, ab) \\
& \vdash (q_0, aaba, abbaZ_0) & // \ \delta(q_0, a, a) \rightarrow (q_1, \varepsilon) \\
& \vdash (q_1, aba, bbaZ_0) & // \text{ no rule to be applied}
\end{aligned}
$$

Syntax Analysis—General View

How to construct a PA?

- Directly
- From a context-free grammar

Syntax Analysis

Virtual reconstruction of a particular derivation tree for the given grammar and sentence of the language.

Note: to make difference between end of the input tape and ignoring the input symbol we will use \$ *to represent end of the input tape (empty input) and* ε *to denote that we do not read the symbol under the reading head and the reading head is not moved.*

Top-Down Analysis

Virtual reconstruction of the derivation tree starts from the starting non-terminal of the respective context-free grammar.

Construction

Let $G = (N, T, P, S)$ be a context-free grammar then construct $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ this way:

- $Q = \{q_0, q, q_F\}$
- $\Sigma = T \cup \{\$\}$
- $\Gamma = \{Z_0\} \cup T \cup N, \quad \{Z_0, \$\} \cap (T \cup N) = \emptyset$
- $F = \{q_F\}$
- For all $A \to \alpha \in P, A \in N, \alpha \in (N \cup T)^*$
  add $(q, \varepsilon, A) \to \{(q, \alpha)\}$ to δ
  For all $a \in T$ add $(q, a, a) \to \{(q, \varepsilon)\}$ to δ
  Add $(q_0, \varepsilon, Z_0) \to \{(q, SZ_0)\}$ to δ
  Add $(q, \$, Z_0) \to \{(q_F, \varepsilon)\}$ to δ

Example—Construction

CF Grammar

Let $G = (N, T, P, S)$, where: $N = \{S, E\}$, $T = \{ (,), +, i \}$,
$P = \{ S \to SS \mid E,\ E \to E + E \mid i \mid (E) \}$.

PA Construction

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where:

- $Q = \{q_0, q, q_F\}$, $F = \{q_F\}$
- $\Sigma = T \cup \{\$\} = \{ (,), +, i, \$ \}$
- $\Gamma = \{Z_0\} \cup T \cup N = \{S, E, (,), +, i, Z_0\}$
- $\delta(q, \$, Z_0) \to \{(q_F, \varepsilon)\}$
  $\delta(q_0, \varepsilon, Z_0) \to \{(q, SZ_0)\}$
  $\delta(q, \varepsilon, S) \to \{(q, SS), (q, E)\}$
  $\delta(q, \varepsilon, E) \to \{(q, E + E), (q, i), (q, (E))\}$
  $\delta(q, (, () \to \{(q, \varepsilon)\}$
  $\delta(q,),)) \to \{(q, \varepsilon)\}$
  $\delta(q, +, +) \to \{(q, \varepsilon)\}$
  $\delta(q, i, i) \to \{(q, \varepsilon)\}$

Example—Analysis

$(q_0, i+i(i), Z_0)$	$// \ \delta(q_0, \varepsilon, Z_0) \rightarrow (q, SZ_0)$
$\vdash (q, i+i(i), SZ_0)$	$// \ \delta(q, \varepsilon, S) \rightarrow (q, SS)$
$\vdash (q, i+i(i), SSZ_0)$	$// \ \delta(q, \varepsilon, S) \rightarrow (q, E)$
$\vdash (q, i+i(i), ESZ_0)$	$// \ \delta(q, \varepsilon, E) \rightarrow (q, E+E)$
$\vdash (q, i+i(i), E+ESZ_0)$	$// \ \delta(q, \varepsilon, E) \rightarrow (q, i)$
$\vdash (q, i+i(i), i+ESZ_0)$	$// \ \delta(q, i, i) \rightarrow (q, \varepsilon)$
$\vdash (q, +i(i), +ESZ_0)$	$// \ \delta(q, +, +) \rightarrow (q, \varepsilon)$
$\vdash (q, i(i), ESZ_0)$	$// \ \delta(q, \varepsilon, E) \rightarrow (q, i)$
$\vdash (q, i(i), iSZ_0)$	$// \ \delta(q, i, i) \rightarrow (q, \varepsilon)$
$\vdash (q, (i), SZ_0)$	$// \ \delta(q, \varepsilon, S) \rightarrow (q, E)$
$\vdash (q, (i), EZ_0)$	$// \ \delta(q, \varepsilon, E) \rightarrow (q, (E))$
$\vdash (q, (i), (E)Z_0)$	$// \ \delta(q, (, () \rightarrow (q, \varepsilon)$
$\vdash (q, i), E)Z_0)$	$// \ \delta(q, \varepsilon, E) \rightarrow (q, i)$
$\vdash (q, i), i)Z_0)$	$// \ \delta(q, i, i) \rightarrow (q, \varepsilon)$
$\vdash (q,),)Z_0)$	$// \ \delta(q,),)) \rightarrow (q, \varepsilon)$
$\vdash (q, \$, Z_0)$	$// \ \delta(q, \$, Z_0) \rightarrow (q_F, \varepsilon)$
$\vdash (q_F, \$, \varepsilon)$	

Example—Derivation Reconstruction

Grammar “rules” used in the following order:

1. $S \rightarrow SS$
2. $S \rightarrow E$
3. $E \rightarrow E + E$
4. $E \rightarrow i$
5. $E \rightarrow i$
6. $S \rightarrow E$
7. $E \rightarrow (E)$
8. $E \rightarrow i$

It was **left derivation**, thus we get:

$$S \Rightarrow SS \Rightarrow ES \Rightarrow E + ES \Rightarrow i + ES \Rightarrow i + iS \Rightarrow$$
$$\Rightarrow i + iE \Rightarrow i + i(E) \Rightarrow i + i(i)$$

Definitions

Bottom-Up Analysis

Reconstruction of the derivation tree starts from the symbols of the input sentence up to the starting non-terminal of the respective grammar.

Definition of Extended PA

An extended pushdown automaton is a seven-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where Q, Σ, Γ, q_0, Z_0, and F have the same meaning as for pushdown automaton.
δ is defined as mapping $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^* \rightarrow 2^{Q \times \Gamma^*}$

Construction

Let $G = (N, T, P, S)$ be a context-free grammar then construct $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ this way:

- $Q = \{q_0, q_1, q_F\}$
- $\Sigma = T \cup \{\$\}$
- $\Gamma = \{Z_0\} \cup T \cup N$
- $F = \{q_F\}$
- For all $A \to \alpha \in P, A \in N, \alpha \in (N \cup T)^*$
  add $(q_0, \varepsilon, \alpha^R) \to \{(q_0, A)\}$ to δ
  For all $a \in T$ and all $z \in \Gamma$ add $(q_0, a, z) \to \{(q_0, az)\}$ to δ
  Add $(q_0, \varepsilon, S) \to \{(q_1, \varepsilon)\}$ to δ
  Add $(q_1, \$, Z_0) \to \{(q_F, \varepsilon)\}$ to δ

Example—Construction

CF Grammar

Let $G = (N, T, P, S)$, where: $N = \{S, E\}$, $T = \{ \,(,), +, i\, \}$,
$P = \{ \; S \rightarrow SS \mid E, \;\; E \rightarrow E + E \mid i \mid (E) \; \}$.

PA Construction

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where:

- $Q = \{q_0, q_1, q_F\}$, $F = \{q_F\}$
- $\Sigma = T \cup \{\$\} = \{ \,(,), +, i, \$\, \}$
- $\Gamma = \{Z_0\} \cup T \cup N = \{S, E, (,), +, i, Z_0\}$
- $\delta(q_0, \varepsilon, S) \rightarrow \{(q_1, \varepsilon)\}$
  $\delta(q_1, \$, Z_0) \rightarrow \{(q_F, \varepsilon)\}$
  $\delta(q_0, \varepsilon, SS) \rightarrow \{(q_0, S)\}$
  $\delta(q_0, \varepsilon, E) \rightarrow \{(q_0, S)\}$
  $\delta(q_0, \varepsilon, E + E) \rightarrow \{(q_0, E)\}$
  $\delta(q_0, \varepsilon, i) \rightarrow \{(q_0, E)\}$
  $\delta(q_0, \varepsilon,)E() \rightarrow \{(q_0, E)\}$
  $\ldots$

Example—Construction

PA Construction

- ...

  $\delta(q_0, (, Z_0) \rightarrow \{(q_0, (Z_0)\}$
  $\delta(q_0,), Z_0) \rightarrow \{(q_0,)Z_0)\}$
  $\delta(q_0, +, Z_0) \rightarrow \{(q_0, +Z_0)\}$
  $\delta(q_0, i, Z_0) \rightarrow \{(q_0, iZ_0)\}$
  $\delta(q_0, (, S) \rightarrow \{(q_0, (S)\}$
  $\delta(q_0,), S) \rightarrow \{(q_0,)S)\}$
  $\delta(q_0, +, S) \rightarrow \{(q_0, +S)\}$
  $\delta(q_0, i, S) \rightarrow \{(q_0, iS)\}$
  $\delta(q_0, (, E) \rightarrow \{(q_0, (E)\}$
  $\delta(q_0,), E) \rightarrow \{(q_0,)E)\}$
  $\delta(q_0, +, E) \rightarrow \{(q_0, +E)\}$
  $\delta(q_0, i, E) \rightarrow \{(q_0, iE)\}$
  ...

... for all $z \in \Gamma$

Example—Analysis

$$
\begin{aligned}
&(q_0, i+i(i), Z_0) && // \ \delta(q_0, i, Z_0) \to (q_0, iZ_0)\\
&\vdash (q_0, +i(i), iZ_0) && // \ \delta(q_0, \varepsilon, i) \to (q_0, E)\\
&\vdash (q_0, +i(i), EZ_0) && // \ \delta(q_0, +, E) \to (q_0, +E)\\
&\vdash (q_0, i(i), +EZ_0) && // \ \delta(q_0, i, +) \to (q_0, i+)\\
&\vdash (q_0, (i), i+EZ_0) && // \ \delta(q_0, \varepsilon, i) \to (q_0, E)\\
&\vdash (q_0, (i), E+EZ_0) && // \ \delta(q_0, \varepsilon, E+E) \to (q_0, E)\\
&\vdash (q_0, (i), EZ_0) && // \ \delta(q_0, \varepsilon, E) \to (q_0, S)\\
&\vdash (q_0, (i), SZ_0) && // \ \delta(q_0, (, S) \to (q_0, (S)\\
&\vdash (q_0, i), (SZ_0) && // \ \delta(q_0, i, () \to (q_0, i()\\
&\vdash (q_0,), i(SZ_0) && // \ \delta(q_0, \varepsilon, i) \to (q_0, E)\\
&\vdash (q_0,), E(SZ_0) && // \ \delta(q_0,), E) \to (q_0,)E)\\
&\vdash (q_0, \$,)E(SZ_0) && // \ \delta(q_0, \varepsilon,)E() \to (q_0, E)\\
&\vdash (q_0, \$, ESZ_0) && // \ \delta(q_0, \varepsilon, E) \to (q_0, S)\\
&\vdash (q_0, \$, SSZ_0) && // \ \delta(q_0, \varepsilon, SS) \to (q_0, S)\\
&\vdash (q_0, \$, SZ_0) && // \ \delta(q_0, \varepsilon, S) \to (q_1, \varepsilon)\\
&\vdash (q_1, \$, Z_0) && // \ \delta(q_1, \$, Z_0) \to (q_F, \varepsilon)\\
&\vdash (q_F, \$, \varepsilon)
\end{aligned}
$$

Example—Derivation Reconstruction

Grammar “rules” used in the following order:

1. $E \rightarrow i$
2. $E \rightarrow i$
3. $E \rightarrow E + E$
4. $S \rightarrow E$
5. $E \rightarrow i$
6. $E \rightarrow (E)$
7. $S \rightarrow E$
8. $S \rightarrow SS$

It was **right derivation** in reverse order, thus we get:

$$S \Rightarrow SS \Rightarrow SE \Rightarrow S(E) \Rightarrow S(i) \Rightarrow E(i) \Rightarrow$$
$$\Rightarrow E + E(i) \Rightarrow E + i(i) \Rightarrow i + i(i)$$

Importance of Determinism

Non-deterministic analysis of context-free languages

- Possible
- With backtracking
- Inefficient

Deterministic analysis of context-free languages

- Possible for certain, but sufficiently large subset of context-free languages
- Without backtracking
- Efficient

Non-Deterministic Pushdown Automata

Presented definition matches non-deterministic pushdown automata

Where is the non-determinism?

- $\delta(q, a, z) \rightarrow \{(q', \alpha), (q'', \beta), \dots\}$ — multiple choice
- $\delta(q, \varepsilon, z) \rightarrow \{\dots\}$ — ε matches always

All examples presented so far demonstrated non-deterministic pushdown automata

Deterministic Pushdown Automata

Formal Definition of Deterministic Pushdown Automaton

A deterministic pushdown automaton is a seven-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where

- Q — finite *set of states*
- Σ — finite *input alphabet*
- Γ — finite *alphabet of pushdown symbols*
- δ — mapping $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow Q \times \Gamma^*$ *transition function*
- $q_0 \in Q$ — *starting/initial state*
- $Z_0 \in \Gamma$ — *start symbol* on the pushdown
- $F \subseteq Q$ — *set of final states*

Configuration, move, final configuration, and accepted language defined the same way

Deterministic Pushdown Automata

Is the definition sufficient to ensure determinism?

No!

- Eliminated multiple choices,
- but the other option must be eliminated by **proper construction.**
  Compare: $\delta(q, a, Z) \rightarrow (p, u)$ versus $\delta(q, \varepsilon, Z) = (p', u')$

Deterministic Pushdown Automata

Is it possible to find deterministic automaton for every non-deterministic one?

Unfortuntely, **no!**

- Principal reasons
- Ambiguity of grammar used for automata construction

Let us try . . .

Example 1' ($a^n b^n$, $n \geq 0$) — Introduction

Possible solution features:

- Separation of empty and non-empty input cases
- State for reading of *a*s with duplication to the pushdown
- State for reading of *b*s when removing symbols from pushdown to check that number of both is equal
- Separete final state — final configuration 3 (empty pushdown & final state)

Example 1' ($a^n b^n$, $n \geq 0$) — Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0, q_1, q_2, q_F\}$
$\Sigma = \{a, b, \$\}$
$\Gamma = \{Z_0\}$
$F = \{q_F\}$

$$
\begin{aligned}
\delta\colon\ & \delta(q_0, a, Z_0) \to (q_1, Z_0Z_0) \\
& \delta(q_1, a, Z_0) \to (q_1, Z_0Z_0) \\
& \delta(q_1, b, Z_0) \to (q_2, \varepsilon) \\
& \delta(q_2, b, Z_0) \to (q_2, \varepsilon) \\
& \delta(q_2, \$, Z_0) \to (q_F, \varepsilon) \\
& \delta(q_0, \$, Z_0) \to (q_F, \varepsilon)
\end{aligned}
$$

Example 1' ($a^n b^n$, $n \geq 0$) — Acceptance

$$
\begin{array}{ll}
(q_0, aaabbb, Z_0) & // \ \delta(q_0, a, Z_0) \rightarrow (q_1, Z_0 Z_0) \\
\quad \vdash (q_1, aabbb, Z_0 Z_0) & // \ \delta(q_1, a, Z_0) \rightarrow (q_1, Z_0 Z_0) \\
\quad \vdash (q_1, abbb, Z_0 Z_0 Z_0) & // \ \delta(q_1, a, Z_0) \rightarrow (q_1, Z_0 Z_0) \\
\quad \vdash (q_1, bbb, Z_0 Z_0 Z_0 Z_0) & // \ \delta(q_1, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, bb, Z_0 Z_0 Z_0) & // \ \delta(q_2, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, b, Z_0 Z_0) & // \ \delta(q_2, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, \$, Z_0) & // \ \delta(q_2, \$, Z_0) \rightarrow (q_F, \varepsilon) \\
\quad \vdash (q_F, \$, \varepsilon) &
\end{array}
$$

Example 1' ($a^n b^n$, $n \geq 0$) — Failure

$$
\begin{array}{ll}
(q_0, aabbb, Z_0) & // \ \delta(q_0, a, Z_0) \rightarrow (q_1, Z_0 Z_0) \\
\quad \vdash (q_1, abbb, Z_0 Z_0) & // \ \delta(q_1, a, Z_0) \rightarrow (q_1, Z_0 Z_0) \\
\quad \vdash (q_1, bbb, Z_0 Z_0 Z_0) & // \ \delta(q_1, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, bb, Z_0 Z_0) & // \ \delta(q_2, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, b, Z_0) & // \ \delta(q_2, b, Z_0) \rightarrow (q_2, \varepsilon) \\
\quad \vdash (q_2, \$, \varepsilon) & // \text{ no rule to be applied}
\end{array}
$$

Example 1' ($a^n b^n$, $n \geq 0$) — Discussion

Creation possible

No algorithm to transform the non-deterministic PA

Other solutions may work as well

What modification can be done?

Can we reduce number of states?

A non-ambiguous grammar exists

$S \rightarrow \varepsilon \mid aSb$

Other features: LL(1)

Example 2' ($ww^R, w \in \{a, b\}^+$) — Discussion

No deterministic PA exists

Reason: it is impossible to denote the midpoint without additional information deterministically.

A non-ambiguous grammar exists though

$S \rightarrow \varepsilon \mid aSa \mid bSb$

Other features: can't be LL(1)

Question: can a deterministic PA be defined for language $wcw^R, w \in \{a, b\}^+$?

Example 3 — Introduction

Definition based on grammar description of the language:

$P = \{ S \rightarrow SS \mid E,\ E \rightarrow E + E \mid i \mid (E) \}$.

Problem: ambiguous grammar

Consider: $i + i + i$

$S \Rightarrow E \Rightarrow E_1 + E_2 \Rightarrow E_{1.1} + E_{1.2} + E_2 \Rightarrow \ldots$

$S \Rightarrow E \Rightarrow E_1 + E_2 \Rightarrow E_1 + E_{2.1} + E_{2.2} \Rightarrow \ldots$

We can make it non-ambiguous:

$P = \{S \rightarrow ES',\ S' \rightarrow \varepsilon \mid ES',\ E \rightarrow (E)E' \mid iE',\ E' \rightarrow \varepsilon \mid +E\}$

Does it help?

Not for human... but for machine, maybe.

We need a modification of PA.

Modifications of Pushdown Automata

Purpose:

- new features to make our life easier
- limit features to study properties and simplify analysis
- extend features (regulated PA)

Selected modifications:

- Non-Head-Moving PA
- One-Turn PA
- Atomic PA

Head Does Not Move During Reading

Limiting feature: when recognizing symbol under the reading head, the head has to move.

Formal modification of transition function:
$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow Q \times \Gamma^* \times \{\mathcal{R}, \mathcal{M}\}$
$\mathcal{R}$ stays for reading the symbol under the reading head only
$\mathcal{M}$ stays for read&move (traditional behaviour)

Notation:
$\delta(q, a, z) \rightarrow (q', \alpha)_R$
$\delta(q, a, z) \rightarrow (q', \alpha)_M$ we write $\delta(q, a, z) \rightarrow (q', \alpha)$ if clear
where $q, q' \in Q$, $a \in \Sigma$, $z \in \Gamma$, $\alpha \in \Gamma^*$

Generally non-deterministic variant:
$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times \Gamma^* \times \{\mathcal{R}, \mathcal{M}\}}$

Example 1” ($a^n b^n$, $n \geq 0$) — Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0\}$
$\Sigma = \{a, b, \$\}$
$\Gamma = \{a, b, Z_0\}$
$F = \{q_0\}$

δ: $\delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R$
$\delta(q_0, b, Z_0) \rightarrow (q_0, \varepsilon)_R$
$\delta(q_0, a, a) \rightarrow (q_0, \varepsilon)$
$\delta(q_0, b, b) \rightarrow (q_0, \varepsilon)$
$\delta(q_0, \$, Z_0) \rightarrow (q_0, \varepsilon)$

Final configuration 3.

Example 1” ($a^n b^n$, $n \geq 0$) — Acceptance

$$
\begin{array}{ll}
(q_0, aaabbb, Z_0) & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R \\
\quad \vdash (q_0, aaabbb, aZ_0b) & // \ \delta(q_0, a, a) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, aabbb, Z_0b) & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R \\
\quad \vdash (q_0, aabbb, aZ_0bb) & // \ \delta(q_0, a, a) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, abbb, Z_0bb) & // \ \delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R \\
\quad \vdash (q_0, abbb, aZ_0bbb) & // \ \delta(q_0, a, a) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, bbb, Z_0bbb) & // \ \delta(q_0, b, Z_0) \rightarrow (q_0, \varepsilon)_R \\
\quad \vdash (q_0, bbb, bbb) & // \ \delta(q_0, b, b) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, bb, bb) & // \ \delta(q_0, b, b) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, b, b) & // \ \delta(q_0, b, b) \rightarrow (q_0, \varepsilon) \\
\quad \vdash (q_0, \$, \varepsilon) &
\end{array}
$$

Example 1” ($a^n b^n$, $n \geq 0$) — Failure

$(q_0, aabbb, Z_0)$	// $\delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R$
$\vdash (q_0, aabbb, aZ_0b)$	// $\delta(q_0, a, a) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, abbb, Z_0b)$	// $\delta(q_0, a, Z_0) \rightarrow (q_0, aZ_0b)_R$
$\vdash (q_0, abbb, aZ_0bb)$	// $\delta(q_0, a, a) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, bbb, Z_0bb)$	// $\delta(q_0, b, Z_0) \rightarrow (q_0, \varepsilon)_R$
$\vdash (q_0, bbb, bb)$	// $\delta(q_0, b, b) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, bb, b)$	// $\delta(q_0, b, b) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, b, \varepsilon)$	// no rule to be applied

Example 3' — Definition

$$P = \{S \to ES',\ S' \to \varepsilon \mid ES',\ E \to (E)E' \mid iE',\ E' \to \varepsilon \mid +E\}$$

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0\}$
$\Sigma = \{+, i, (,), \$\}$
$\Gamma = \{+, i, (,), E, S', E', Z_0\}$
$F = \{q_0\}$

δ: $\delta(q_0, (, Z_0) \to (q_0, ES')_R$
$\delta(q_0, i, Z_0) \to (q_0, ES')_R$
$\delta(q_0, (, E) \to (q_0, (E)E')_R$
$\delta(q_0, i, E) \to (q_0, iE')_R$
$\delta(q_0, (, S') \to (q_0, ES')_R$
$\delta(q_0, i, S') \to (q_0, ES')_R$
$\delta(q_0, \$, S') \to (q_0, \varepsilon)_R$
...

Example 3' — Definition

...

$$\delta(q_0, +, E') \rightarrow (q_0, +E)_R$$
$$\delta(q_0,), E') \rightarrow (q_0, \varepsilon)_R$$
$$\delta(q_0, (, E') \rightarrow (q_0, \varepsilon)_R$$
$$\delta(q_0, i, E') \rightarrow (q_0, \varepsilon)_R$$
$$\delta(q_0, \$, E') \rightarrow (q_0, \varepsilon)_R$$
$$\delta(q_0,),)) \rightarrow (q_0, \varepsilon)$$
$$\delta(q_0, (, () \rightarrow (q_0, \varepsilon)$$
$$\delta(q_0, i, i) \rightarrow (q_0, \varepsilon)$$
$$\delta(q_0, +, +) \rightarrow (q_0, \varepsilon)$$

Example 3' — Analysis

$$
\begin{array}{ll}
(q_0, i+i(i), Z_0) & // \ \delta(q_0, i, Z_0) \to (q_0, ES')_R \\
\vdash (q_0, i+i(i), ES') & // \ \delta(q_0, i, E) \to (q_0, iE')_R \\
\vdash (q_0, i+i(i), iE'S') & // \ \delta(q_0, i, i) \to (q_0, \varepsilon) \\
\vdash (q_0, +i(i), E'S') & // \ \delta(q_0, +, E') \to (q_0, +E)_R \\
\vdash (q_0, +i(i), +ES') & // \ \delta(q_0, +, +) \to (q_0, \varepsilon) \\
\vdash (q_0, i(i), ES') & // \ \delta(q_0, i, E) \to (q_0, iE')_R \\
\vdash (q_0, i(i), iE'S') & // \ \delta(q_0, i, i) \to (q_0, \varepsilon) \\
\vdash (q_0, (i), E'S') & // \ \delta(q_0, (, E') \to (q_0, \varepsilon)_R \\
\vdash (q_0, (i), S') & // \ \delta(q_0, (, S') \to (q_0, ES')_R \\
\vdash (q_0, (i), ES') & // \ \delta(q_0, (, E) \to (q_0, (E)E')_R \\
\vdash (q_0, (i), (E)E'S') & // \ \delta(q_0, (, () \to (q_0, \varepsilon) \\
\vdash (q_0, i), E)E'S') & // \ \delta(q_0, i, E) \to (q_0, iE')_R \\
\vdash (q_0, i), iE')E'S') & // \ \delta(q_0, i, i) \to (q_0, \varepsilon) \\
\vdash (q_0,), E')E'S') & // \ \delta(q_0,), E') \to (q_0, \varepsilon)_R \\
\vdash (q_0,),)E'S') & // \ \delta(q_0,),)) \to (q_0, \varepsilon) \\
\vdash (q_0, \$, E'S') & // \ \delta(q_0, \$, E') \to (q_0, \varepsilon)_R \\
\vdash (q_0, \$, S') & // \ \delta(q_0, \$, S') \to (q_0, \varepsilon)_R \\
\vdash (q_0, \$, \varepsilon) &
\end{array}
$$

Example 3' — Discussion

Grammar features:

- Non-ambiguous
- LL(1)
- Not trivial to construct

How could we construct the PA?

- By straightforward reasoning from the grammar —**too difficult!**
- By algorithm
  - See Aho, Sethi, Ullman: Compilers . . . in References
  - LL(1) grammar is an input
  - Non-moving pushdown automaton necessary

Definition of One-Turn PA

Definition

Consider two consecutive moves made by a pushdown automaton, *M*. If during the first move *M* does not shorten its pushdown and during the second move it does, then *M* makes a turn during the second move. A pushdown automaton is *one-turn* if it makes no more than one turn with its pushdown during any computation starting from an initial configuration.

Theorem

One-turn pushdown automata characterize a family of linear languages.
Proof: see Harrison . . . in References.

Discussion

Example of one-turn PA—Example 1'

Can we re-construct Examples 1 and 2 to be one-turn?
YES!

Do it!

Hint: differentiate extension and shortening of pushdown by states, allow just one turn!

Definition of Atomic PA

Definition

Modify δ as a mapping from $Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\})$. During a move, an *atomic* pushdown automaton changes a state and, in addition, performs exactly one of the following actions:

1. it pushes a symbol onto the pushdown
   e.g. $\delta(q, \varepsilon, \varepsilon) \rightarrow \{(q', a)\}$
2. it pops a symbol from the pushdown
   e.g. $\delta(q, \varepsilon, a) \rightarrow \{(q', \varepsilon)\}$
3. it reads an input symbol
   e.g. $\delta(q, a, \varepsilon) \rightarrow \{(q', \varepsilon)\}$

Example 1''' ($a^n b^n$, $n \geq 0$) — Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$Q = \{q_0, q_a, q_{a'}, q_{a''}, q_b, q_F\}$
$\Sigma = \{a, b, \$\}$
$\Gamma = \{b, Z_0\}$
$F = \{q_F\}$

δ: $\delta(q_0, a, \varepsilon) \to (q_a, \varepsilon)$
$\delta(q_a, \varepsilon, Z_0) \to (q_{a'}, \varepsilon)$
$\delta(q_{a'}, \varepsilon, \varepsilon) \to (q_{a''}, b)$
$\delta(q_{a''}, \varepsilon, \varepsilon) \to (q_0, Z_0)$
$\delta(q_0, b, \varepsilon) \to (q_b, \varepsilon)$
$\delta(q_b, \varepsilon, Z_0) \to (q_b, \varepsilon)$
$\delta(q_b, \varepsilon, b) \to (q_0, \varepsilon)$
$\delta(q_0, \$, \varepsilon) \to (q_F, \varepsilon)$
$\delta(q_F, \varepsilon, Z_0) \to (q_F, \varepsilon)$

Features: deterministic, atomic $\Rightarrow$ easy to implement

Example 1''' ($a^n b^n$, $n \geq 0$) — Acceptance

$(q_0, aabb, Z_0)$	// $\delta(q_0, a, \varepsilon) \rightarrow (q_a, \varepsilon)$
$\vdash (q_a, abb, Z_0)$	// $\delta(q_a, \varepsilon, Z_0) \rightarrow (q_{a'}, \varepsilon)$
$\vdash (q_{a'}, abb, \varepsilon)$	// $\delta(q_{a'}, \varepsilon, \varepsilon) \rightarrow (q_{a''}, b)$
$\vdash (q_{a''}, abb, b)$	// $\delta(q_{a''}, \varepsilon, \varepsilon) \rightarrow (q_0, Z_0)$
$\vdash (q_0, abb, Z_0 b)$	// $\delta(q_0, a, \varepsilon) \rightarrow (q_a, \varepsilon)$
$\vdash (q_a, bb, Z_0 b)$	// $\delta(q_a, \varepsilon, Z_0) \rightarrow (q_{a'}, \varepsilon)$
$\vdash (q_{a'}, bb, b)$	// $\delta(q_{a'}, \varepsilon, \varepsilon) \rightarrow (q_{a''}, b)$
$\vdash (q_{a''}, bb, bb)$	// $\delta(q_{a''}, \varepsilon, \varepsilon) \rightarrow (q_0, Z_0)$
$\vdash (q_0, bb, Z_0 bb)$	// $\delta(q_0, b, \varepsilon) \rightarrow (q_b, \varepsilon)$
$\vdash (q_b, b, Z_0 bb)$	// $\delta(q_b, \varepsilon, Z_0) \rightarrow (q_b, \varepsilon)$
$\vdash (q_b, b, bb)$	// $\delta(q_b, \varepsilon, b) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, b, b)$	// $\delta(q_0, b, \varepsilon) \rightarrow (q_b, \varepsilon)$
$\vdash (q_b, \$, b)$	// $\delta(q_b, \varepsilon, b) \rightarrow (q_0, \varepsilon)$
$\vdash (q_0, \$, \varepsilon)$	// $\delta(q_0, \$, \varepsilon) \rightarrow (q_F, \varepsilon)$
$\vdash (q_F, \$, \varepsilon)$	

Example 1''' ($a^n b^n$, $n \geq 0$) — Failure

(q_0, a, Z_0)	// $\delta(q_0, a, \varepsilon) \to (q_a, \varepsilon)$
$\vdash (q_a, \$, Z_0)$	// $\delta(q_a, \varepsilon, Z_0) \to (q_{a'}, \varepsilon)$
$\vdash (q_{a'}, \$, \varepsilon)$	// $\delta(q_{a'}, \varepsilon, \varepsilon) \to (q_{a''}, b)$
$\vdash (q_{a''}, \$, b)$	// $\delta(q_{a''}, \varepsilon, \varepsilon) \to (q_0, Z_0)$
$\vdash (q_0, \$, Z_0 b)$	// $\delta(q_0, \$, \varepsilon) \to (q_F, \varepsilon)$
$\vdash (q_F, \$, Z_0 b)$	// $\delta(q_F, \varepsilon, Z_0) \to (q_F, \varepsilon)$
$\vdash (q_F, \$, b)$	// no rule to be applied

(q_0, b, Z_0)	// $\delta(q_0, b, \varepsilon) \to (q_b, \varepsilon)$
$\vdash (q_b, \$, Z_0)$	// $\delta(q_b, \varepsilon, Z_0) \to (q_b, \varepsilon)$
$\vdash (q_b, \$, \varepsilon)$	// no rule to be applied

Example 1''' ($a^n b^n$, $n \geq 0$) — Discussion

Easy to implement $\not\Leftarrow$ easy to define (algorithmically)

Quite a lot of steps . . .

Automaton without movement of reading head is much more efficient . . .

Importance in the formal theory

- simplicity of definition
- minimality of description
- . . .

Conclusion

We have presented:

- Definition of pushdown automaton
- Connected terms: configuration, transition step, language accepted by PA
- Deterministic and non-deterministic variant
- Some modifications

Main importance in:

- Analysis of context-free languages
- Theoretical computer science

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