The α-Embracing Countour

Manuel Abellanas∗ Mercè Claverol† Inês Matos‡

Abstract

Every notion of depth induces a stratification of the plane in regions of points with the same depth with respect to a given set of points. The boundaries of these regions, also known as depth-contours, are an appropriate tool for data visualization and have already been studied for some depths like Tukey depth [5, 7, 8, 9] and Delaunay depth [3, 6]. The contours also have applications in quality illumination as in the case of good α-illumination [2]. The first α-depth contour is also known as the α-embracing contour. We prove that the first α-depth contour has linear size and we give an algorithm to compute it in $O(n^2)$ running time and $O(n)$ space.

1 Introduction and Related Work

Data depth has been considered as a measure to check how deep or central a given point is with respect to a multivariate distribution. A notion of depth induces a stratification of the plane in regions of points that share their depth with respect to a given set of points. The boundaries of these regions are called depth-contours and they are used as a tool for data visualization since they provide a quick and informative overview of the shape and properties of the point set. Several different notions of depth have been studied, for example, the location depth (also known by halfspace depth or Tukey depth [9]) and Delaunay depth [6]. The Tukey depth measures how many points of a given set can be separated from a point $q$ by means of a half-plane. One can consider other ways to separate a point, choosing the best to fit a certain application. Next we define the depth used in this paper, the α-depth.

Definition 1.1. Let $S$ be a set of $n$ points in the plane. A point $q$ in the plane has α-depth $k$ with respect to $S$ if every open wedge of angle $\alpha$, with apex at $q$, contains at least $k$ points of $S$ and there is at least one such wedge containing exactly $k$ points.

![Figure 1](image.png)

Figure 1: (a) Point $q$ has Tukey depth equal to 3 and π-depth equal to 2 because we are considering open wedges. (b) Point $q$ has α-depth equal to 1. (c) The α-arc of $ab$ associated to the line segment $s_k s_{k+1}$, where $s_k s_{k+1}$ is an edge of the CH($S$).

If we were to consider the points on the border of the wedge, then the α-depth would be a generalization of the Tukey depth. The wedge would be seen as a way to separate $q$ instead of a half-plane.

∗Facultad de Informática, Universidad Politécnica de Madrid, supported by grant TIC2003-08933-C02-01, MEL-HP2005-0137 and partially supported by CAMP-DPI-000235-0505 mabellanas@fi.upm.es
†Dep. de Matemática Aplicada IV, Universitat Politècnica de Catalunya, partially supported by projects MEC MTM2006-01267 and Gen. Cat. 2005SGR00692 merc@ma4.upc.edu
‡Dep. de Matemática & CEOC, Universidade de Aveiro, supported by CEOC through Programa POCTI, FCT, co-financed by EC fund FEDER, by Acção No. E-77/06 and supported by a FCT fellowship, grant SFRH/BD/28652/2006 ipmatos@mat.ua.pt
For example, considering closed $\alpha$-wedges for $\alpha = \pi$, the $\alpha$-depth corresponds to the Tukey depth (see Figure 1(a)). However, for the purpose of this paper, this definition considers open $\alpha$-wedges (see Figure 1(b)). A main concern in current theoretical research on data depth is to compute depth contours. The Tukey depth contours have been studied and calculated in [5, 7, 8]. The Delaunay depth contours have also been studied in [3]. The contours have applications in quality illumination too, for example, they can be applied to good illumination [1, 2] as we can see in the following. In this paper, we focus on the construction of the first $\alpha$-depth contour that is also known as the $\alpha$-embracing contour.

The good $\alpha$-illumination [2] is a generalization of the 1-good illumination [1]. A point $p$ in the plane is well $\alpha$-illuminated by a set $S$ of $n$ point lights if there is, at least, one light interior to each wedge centered at $p$ with a given angle $\alpha$. According to the definition of $\alpha$-depth, points well $\alpha$-illuminated have $\alpha$-depth greater or equal to 1. In the 1-good illumination, the convex hull separates the points that are 1-well illuminated from those which are not. The border of the convex hull is a contour that acts as a barrier between the 1-well illuminated points and the rest. A similar structure for the good $\alpha$-illumination is called the $\alpha$-embracing contour or the first $\alpha$-depth contour and is defined next.

**Definition 1.2.** Let $S$ be a set of $n$ points in the plane and $\alpha$ a given angle ($0 \leq \alpha \leq \pi$). The $\alpha$-embracing contour of $S$ is the border of the region of points in the plane which are well $\alpha$-illuminated.

We are going to prove that the $\alpha$-embracing contour is composed by arcs of circumferences. The $\alpha$-arcs defined in the following definition are the key to the $\alpha$-embracing contour construction.

**Definition 1.3.** Let $S$ be a set of $n$ points in the plane, $p$ and $q$ two consecutive vertices of the convex hull of $S$ and $a, b \in S$. The locus of points in the plane that see the line segment $ab$ with a given angle $\alpha$ is composed by two arcs of circumferences with extreme points $a$ and $b$. If $\overline{ab}$ is not perpendicular to $\overline{pq}$, we call one of these two arcs the $\alpha$-arc of $\overline{ab}$ with respect to $\overline{pq}$: the one which is above the line containing $a$ and $b$, if we consider a coordinate system in which $p$ and $q$ belong to the $x$-axis and the points of $S$ have non negative $y$-coordinate. Such arc will be denoted by $\hat{ab}$.

There is an example of Definition 1.3 in Figure 1(c). In this paper we will prove that the $\alpha$-embracing contour consists on a linear number of pieces of $\alpha$-arcs. There is an example of this structure in Figure 2(b) and it also proves the next property.

![Figure 2](image_url) (a) Only the points in the two grey areas are well $\frac{\pi}{2}$-illuminated. (b) The $\frac{\pi}{2}$-embracing contour of the set $S$ which is not connected.

**Proposition 1.4.** The $\alpha$-embracing contour of a set $S$ is not necessarily connected.

In the next section we study the $\alpha$-embracing contour. We prove that it has linear size and show how to compute it in quadratic time and linear space.

## 2 Construction of the $\alpha$-embracing contour

This section is devoted to the construction of the first $\alpha$-depth contour that will be called $\alpha$-embracing contour from now on. Let $S$ be a set of $n$ points in the plane and $\alpha$ a given angle. The construction of the $\alpha$-embracing contour is based on the next lemmas.

**Lemma 2.1.** If two $\alpha$-arcs associated with the same convex hull edge have a common extreme point $s_i \in S$ and intersect at $v \neq s_i$ then $v$ is collinear with the other two extremes of the $\alpha$-arcs.
Proof. Every point on an $\alpha$-arc is the apex of an $\alpha$-wedge such that each ray goes through an extreme point of the corresponding arc. If two $\alpha$-arcs associated with the same convex hull edge have a common extreme point $si \in S$ and intersect at $v \neq sj$ (see Figure 3 (a)), then the wedge with apex at $v$ and one ray through $s_i$ and the other through any of the other extreme points of the two $\alpha$-arcs is an $\alpha$-wedge. Then $v$ and the extreme points of the $\alpha$-arcs different from $s_i$ are collinear.

Figure 3: (a) The $\alpha$-arcs $s_is_j$ and $s_j s_k$ intersect at $v$ and $v$, $s_j$ and $s_k$ are collinear. (b) The wedge $w$ has an angle bigger than $\alpha$ only when $m_j$ is an interior point to $w$. (c) The pieces of the $\alpha$-arcs from $m_i$ to an extreme of $M_{m_i}^l$ up to the first intersection with another arc are ordered according to those extremes.

Let $M_{s_is_{k+1}} = \{m_1, \ldots, m_s\}$ be a set of unoriented $\alpha$-maxima points of $S$ (points of $S$ that allow an empty $\alpha$-wedge centered at them and can be computed by an algorithm by Avis et. al [4]) enclosed in the region delimited by the line segment $s_is_{k+1}$ and the $\alpha$-arc $s_k s_{k+1}$, where $m_1 = s_k$ and $m_s = s_{k+1}$. For every point $m_i$ consider the perpendicular line to $s_is_{k+1}$ through $m_i$ and the partition of the set $M_{s_is_{k+1}} \setminus \{m_i\}$ in $M_{m_i}^l$ and $M_{m_i}^r$. The set $M_{m_i}^l$ ($M_{m_i}^r$) contains the points on the left (right) of the perpendicular line. Let $m_j^l \in M_{m_i}^l$ be the first point around $m_i$ in clockwise order and $m_j^r \in M_{m_i}$ the last.

Lemma 2.2. Let $m_i \in M_{s_is_{k+1}}$ be a vertex of the $\alpha$-embracing contour. From this vertex, the following pieces of the $\alpha$-arcs that are part of the $\alpha$-embracing contour are:

1. the piece of the $\alpha$-arc $\widehat{m_im_j^l}$ from $m_i$ ($i \neq 1$) up to the first intersection with another $\alpha$-arc that has one of its extreme points in $M_{m_i}^l$.

2. the piece of the $\alpha$-arc $\widehat{m_im_j^r}$ from $m_i$ ($i \neq s$) up to the first intersection with another $\alpha$-arc that has one of its extreme points in $M_{m_i}^r$.

Proof. Let $m_j$ and $m_j'$ be two points of $M_{m_i}^l$, $m_j$ is first in clockwise order around $m_i$ (similar arguments hold for two points of $M_{m_i}^r$).

Any wedge $w$ with apex at a point of $\widehat{m_im_j}$ and with one ray through $m_i$ and the other through $m_j'$ has an angle bigger than $\alpha$ if and only if that wedge contains $m_j$ in its interior. This situation happens when the wedge apex and $m_i$ are in the same half-plane defined by the support line through $\widehat{m_jm_j'}$ (see Figure 3(b)). That is, since $m_j$, $m_j'$ and $\widehat{m_jm_j'}$ are collinear by Lemma 2.1, if the apex of a wedge lies at a point of the arc $\widehat{m_jm_j}$ from $m_i$ up to $\widehat{m_jm_j'}$ and it has one ray through $m_i$ and the other through $m_j'$, its angle is bigger than $\alpha$. Therefore the apex of an $\alpha$-wedge with a ray through $m_i$ and the other through $m_j'$ must be out of the region enclosed by $\widehat{m_jm_j'}$ and $\widehat{m_jm_j}$.

If there is not an intersection $\widehat{m_im_j}$ and $\widehat{m_im_j'}$, the arc $\widehat{m_im_j}$ appears first in the clockwise order around $m_i$. The same happens when the two arcs intersect, $\widehat{m_im_j}$ is still the first to appear from $m_i$ up to their intersection point. If another $\alpha$-arc not having $m_i$ as an extreme point cuts $\widehat{m_jm_j'}$ and $\widehat{m_im_j}$ on points of the pieces of the $\alpha$-arcs from $m_i$ up to $\widehat{m_jm_j'}$, then the pieces from $m_i$ up to this new cut have the same order than the previous pieces.

The pieces of the $\alpha$-arcs from $m_i$ to an extreme of the same subset ($M_{m_i}^l$, or $M_{m_i}^r$) up to the first intersection with some arc are ordered according to those extremes (see Figure 3(c)). Then if $m_i$
is a vertex of the $\alpha$-embracing contour, only the pieces of the arcs $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ are part of the embracing contour. The pieces of these two arcs from $m_i$ are only part of the embracing contour until they intersect with another arc with an extreme in the same set ($M_{m_i}^L$ for $\widehat{m_i m_i'}$ and $M_{m_i}^R$ for $\widehat{m_i m_i''}$). This is only true up to the first intersection because the order of the arcs swaps there.

Let $L_{s_k s_{k+1}}$ be the set of lines connecting $m_i$ and $m_i'$ and $m_i$ and $m_i''$, for every point $m_i \in M_{s_k s_{k+1}}$.

**Lemma 2.3.** A vertex of the $\alpha$-embracing contour is either a point of a line of $L_{s_k s_{k+1}}$ or an intersection point between $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ for some $m_i \in M_{s_k s_{k+1}}$. Furthermore, every line of $L_{s_k s_{k+1}}$ contains one vertex of the first contour that does not belong to $M_{s_k s_{k+1}}$.

**Proof.** First we prove that all the vertices of the $\alpha$-embracing contour lie on a line of $L_{s_k s_{k+1}}$ or they are an intersection between $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ for some $m_i \in M_{s_k s_{k+1}}$.

![Figure 4: In both figures the $\alpha$-embracing contour is represented in a solid trace. (a) The point $m_i$ is not a vertex of the $\alpha$-embracing contour but the intersection of the arcs $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ is. There are two other vertices that lie on two lines of $L_{s_k s_{k+1}}$ and are the result of the intersection of the two previous arcs with the arc $\widehat{m_i m_i'}$. (b) The point $m_i$ is a vertex of the $\alpha$-embracing contour and there are two other vertices that lie on two lines of $L_{s_k s_{k+1}}$ and are the result of the intersection of the two previous arcs with the arc $\widehat{m_i m_i''}$.](image)

By the definition of the set $L_{s_k s_{k+1}}$, all the points of $M_{s_k s_{k+1}}$ lie on some line of the set $L_{s_k s_{k+1}}$ and they are possible vertices of the first contour since they are unoriented $\alpha$-maxima [4]. If the $\alpha$-arcs $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ intersect each other, that means that there is an empty $\alpha$-wedge whose apex is at one of the arcs from $m_i$ up to some intersection point, $v_f$ of $v_l$, with some other arc ($v_f$ for the arc with the extreme point $m_i'$ and $v_l$ for the other). The $\alpha$-wedge apex can only be located at any piece of the arc $\widehat{m_i m_i'}$ ($\widehat{m_i m_i''}$) from $m_i$ to $v_f$ ($v_l$). The $\alpha$-wedge is empty when its apex is located at the intersection of both arcs, $v_i$. If the apex moves upwards along $\overrightarrow{m_i v_i}$ then one of the extremes of the arcs will become an interior point of the $\alpha$-wedge. So the point where the apex is located no longer is a point of the $\alpha$-embracing contour. If the apex moves on the opposite direction, then the empty wedge has an angle bigger than $\alpha$, so $v_i$ is a vertex of the embracing contour. By Lemma 2.2, the arcs $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ are also part of the embracing contour (see Figure 4).

Suppose that there are other vertices of the $\alpha$-embracing contour different from the points of $M_{s_k s_{k+1}}$ and that are not intersections between $\widehat{m_i m_i'}$ and $\widehat{m_i m_i''}$ for some $m_i \in M_{s_k s_{k+1}}$. We will see that these vertices are points of some line of $L_{s_k s_{k+1}}$ and are located at the intersection of the $\alpha$-arcs with extreme points of the set $M_{s_k s_{k+1}}$. Moreover these arcs have a common extreme. On the contrary, suppose that the vertex of the $\alpha$-embracing contour is an intersection point between two $\alpha$-arcs without a common extreme. Since these arcs cross, then the $\alpha$-wedge with apex in this intersection point and a ray through one extreme of one of the arcs contains a point. This interior point is an extreme of the other arc. In this case, the $\alpha$-wedge is not empty which contradicts the fact that it is a vertex of the $\alpha$-embracing contour.
Hence we can suppose that a vertex of the $\alpha$-embracing contour is an intersection point between two $\alpha$-arcs with a common extreme. Applying Lemma 2.1, the intersection between these arcs is collinear with the other two extreme points of the arcs. Now we show that one of these two points is the first or the last in clockwise order around the other. In Figure 5, we can see an example where $m'_j$ is the last point in clockwise order around $m_j$ and $m'_i$ is the first in clockwise order around $m_i$. Otherwise, consider the $\alpha$-wedge apex at the vertex of the embracing contour (the intersection point of the arcs). One of its rays goes through the common extreme of the arcs and the other goes through the two extremes that are collinear with the vertex. If one of the extremes were not the first or the last around the other, then it would have to be interior to the $\alpha$-wedge. This contradicts the fact that the $\alpha$-wedge apex is a vertex of the embracing contour. So the vertex lies on a line connecting two points of $M_{sk^k+1}$, where one of them is the last or the first in clockwise order around the other. That is, the vertex lies on a line of $L_{sk^k+1}$.

So now we prove that every line of $L_{sk^k+1}$ contains a vertex of the $\alpha$-embracing contour that does not belong to $M_{sk^k+1}$. Let a line of $L_{sk^k+1}$ be defined by $m_j$ and $m'_j$ (analogously $m_j$ and $m'_i$). Let us compute the cuts between this line and the intersection points between two $\alpha$-arcs with extremes in $M_{sk^k+1}$ (see Figure 6). These cuts are intersection points between pairs of $\alpha$-arcs with a common extreme point, while the other extreme points are $m_j$ and $m'_j$. The first of these two cuts is a vertex of the $\alpha$-embracing contour.

Consider the closest of these cuts to $s_{sk^k+1}$, as long as they are different from $m_j$ and $m'_j$. Next we prove that if we choose this cut to locate the $\alpha$-wedge apex with one ray through the common extreme
of the arcs, $m_i$, and the other through $m_j$ (or $m'_j$ since they are collinear), then this $\alpha$-wedge is empty. Otherwise suppose that $m_k$ is an interior point of the wedge. Note that by definition, $m_k$ cannot be after $m'_j$ in the clockwise order around $m_j$. Then both $\widehat{m_jm_i}$ and $\widehat{m_jm_k}$ have an extreme in $M'_m$. By construction, the $\alpha$-wedge apex is located at the closest cut to $\overline{s_ks_{k+1}}$. The apex is at the intersection between the line $m_jm'_j$ and the intersection between the two $\alpha$-arcs. Then the intersection between the $\alpha$-arcs $\widehat{\overline{m_km_j}}$ and $\widehat{\overline{m_km_j}}$ is located along the line $\overline{m_jm_j}$ by Lemma 2.1 (see Figure 7). However the intersection point of these two $\alpha$-arcs is closer to $\overline{s_ks_{k+1}}$ than the wedge apex. This contradicts the fact that the wedge apex is the closest point to $\overline{s_ks_{k+1}}$. So the $\alpha$-wedge is empty when located at the closest cut to $\overline{s_ks_{k+1}}$. If we move the wedge apex from the intersection point along $\widehat{\overline{m_jm_j}}$ then one extreme point of the arcs becomes an interior point to the wedge. If we move the apex on the opposite direction, then the empty wedge has an angle bigger than $\alpha$. That is, the closest cut to $\overline{s_ks_{k+1}}$ is a vertex of the $\alpha$-embracing contour. So every line of $L_{s_k,s_{k+1}}$ contains a vertex of the $\alpha$-embracing contour that does not belong to $M_{s_k,s_{k+1}}$.

![Figure 7](image-url)

Figure 7: The intersection of the $\alpha$-arcs $\widehat{m_km_j}$ and $\widehat{m_km_j}$ is located along the line $\overline{m_jm_j}$.

![Figure 8](image-url)

Figure 8: Every line of $L_{s_k,s_{k+1}}$ contains a vertex of the $\alpha$-embracing contour. In this example we can see the different kinds of vertices that appear in the $\alpha$-embracing contour.

**Proposition 2.4.** The $\alpha$-embracing contour of a set $S$ of $n$ points has linear complexity.

**Proof.** The $\alpha$-embracing contour restricted to an edge $\overline{s_ks_{k+1}}$ of the CH$(S)$ is a connected chain of $\alpha$-arcs. As consequence of Lemma 2.3, we have seen that the vertices of the $\alpha$-embracing contour restricted to $\overline{s_ks_{k+1}}$ can only be of three kinds: points of $M_{s_k,s_{k+1}}$, intersection points between the first and the last $\alpha$-arcs around some point of $M_{s_k,s_{k+1}}$ or points lying on lines of $L_{s_k,s_{k+1}}$ (see Figure 8). The cardinal of the set $L_{s_k,s_{k+1}}$ is twice the cardinal of the set $M_{s_k,s_{k+1}}$. The union of the sets $M_{s_k,s_{k+1}}$ is the set of all the unoriented $\alpha$-maxima points of $S$, which has $O(n)$ points. So the $\alpha$-embracing contour has linear complexity.

Next we present a quadratic algorithm to compute the $\alpha$-embracing contour based on the previous lemmas.
Algorithm 2.5. Construction of the α-embracing contour of $S$.

**INPUT:** Set of points $S$ in non-degenerate position and an angle $\alpha, 0 < \alpha \leq \pi$.

**OUTPUT:** The α-embracing contour of $S$.

1. Compute the angular order of each point of $S$ with respect to all of the others.
2. Compute CH($S$). Let $\{s_1, \ldots, s_n\} \subseteq S$ be the set of the points on the border of CH($S$).
3. Compute the unoriented $\alpha$-maxima points using the algorithm by Avis et al. [4].
4. For every segment $s_k s_{k+1}$, compute the associated $\alpha$-arc $s_k \hat{s}_{k+1}$.
5. Associate the set of unoriented $\alpha$-maxima points that are in the enclosed region delimited by $s_k s_{k+1}$ and $s_k \hat{s}_{k+1}$ to every line segment $s_k s_{k+1}$. Let $M_{s_k s_{k+1}}$ be that set of points.
6. For every point $m_i \in M_{s_k s_{k+1}}$, consider the first and last points of $M_{s_k s_{k+1}} \setminus \{m_i\}$ in clockwise order around $m_i$. Construct the set of lines $L_{s_k s_{k+1}}$, connecting each point $m_i$ to the first and last points around it. (Note that the last point around $s_k$ is $s_{k+1}$ and the first around $s_{k+1}$ is $s_k$).
7. For every point $m_i \in M_{s_k s_{k+1}}$, construct the first and the last $\alpha$-arcs from $m_i$ up to the first intersection with some line of the set $L_{s_k s_{k+1}}$.
8. If the chain of $\alpha$-arcs just constructed is connected, then it is the $\alpha$-embracing contour of $S$. If the chain of $\alpha$-arcs is disconnected, suppose that the $\alpha$-arc from $m_i$ is disconnected. So this $\alpha$-arc intersects a line of $L_{s_k s_{k+1}}$. Take the point $m_j \in M_{s_k s_{k+1}}$ that lies on that line and is not an extreme point of the previous $\alpha$-arc. Construct the $\alpha$-arc $\hat{m}_i m_j$ up to the first intersection with some other line of the set $L_{s_k s_{k+1}}$ (if it exists). Repeat this process recursively until a connected chain of arcs is achieved.

As a consequence of the preceding paragraphs, we can state the following.

**Theorem 2.6.** The $\alpha$-embracing contour of a set $S$ of $n$ points in the plane can be constructed in $O(n^2)$ time in the worst case and $O(n)$ space.

### 3 Conclusions and Future Work

We introduced the notion of $\alpha$-depth and the first $\alpha$-depth contour, also known as the $\alpha$-embracing contour, as the contour that separates points in the plane that are well $\alpha$-illuminated from the rest. We proved that the $\alpha$-embracing contour consists on a linear number of pieces of $\alpha$-arcs and presented an algorithm to compute the $\alpha$-embracing contour of a point set in the plane that runs in $O(n^2)$ time and $O(n)$ space.

The study of all the $\alpha$-depth contours and their properties is ongoing work, as well as the comparison between the $\alpha$-depth contours and other depth contours.

### 4 Acknowledgments

We wish to thank Ferran Hurtado from the Universitat Politècnica de Catalunya for inspiring us into writing this article and Antonio Bajuelos from the Universidade de Aveiro for his helpful suggestions.
References


