The α-Embracing Countour

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Abstract

Every notion of depth induces a stratification of the plane in regions of points with the same depth with respect to a given set of points. The boundaries of these regions, also known as depth-contours, are an appropriate tool for data visualization and have already been studied for some depths like Tukey depth [5, 7, 8, 9] and Delaunay depth [3, 6]. The contours also have applications in quality illumination as is the case of good α-illumination [2]. The first α-depth contour is also known as the α-embracing contour. We prove that the first α-depth contour has linear size and we give an algorithm to compute it in $\mathcal{O}(n^2)$ running time and $\mathcal{O}(n)$ space.

1 Introduction and Related Work

Data depth has been considered as a measure to check how deep or central a given point is with respect to a multivariate distribution. A notion of depth induces a stratification of the plane in regions of points that share their depth with respect to a given set of points. The boundaries of these regions are called depth-contours and they are used as a tool for data visualization since they provide a quick and informative overview of the shape and properties of the point set. Several different notions of depth have been studied, for example, the location depth (also known by halfspace depth or Tukey depth [9]) and Delaunay depth [6]. The Tukey depth measures how many points of a given set can be separated from a point q by means of a half-plane. One can consider other ways to separate a point, choosing the best to fit a certain application. Next we define the depth used in this paper, the α-depth.

Definition 1.1. Let S be a set of n points in the plane. A point q in the plane has α-depth k with respect to S if every open wedge of angle α, with apex at q, contains at least k points of S and there is at least one such wedge containing exactly k points.

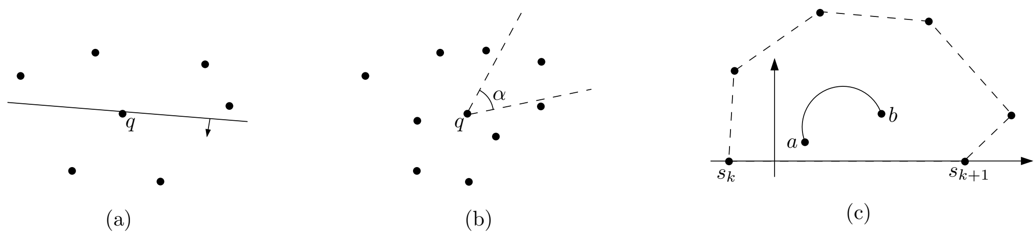

Figure 1: (a) Point q has Tukey depth equal to 3 and π-depth equal to 2 because we are considering open wedges. (b) Point q has α-depth equal to 1. (c) The α-arc of $\overline{ab}$ associated to the line segment $\overline{s_k s_{k+1}}$, where $\overline{s_k s_{k+1}}$ is an edge of the CH(S).

If we were to consider the points on the border of the wedge, then the α-depth would be a generalization of the Tukey depth. The wedge would be seen as a way to separate q instead of a half-plane.

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For example, considering closed α-wedges for $\alpha = \pi$, the α-depth corresponds to the Tukey depth (see Figure 1(a)). However, for the purpose of this paper, this definition considers open α-wedges (see Figure 1(b)). A main concern in current theoretical research on data depth is to compute depth contours. The Tukey depth contours have been studied and calculated in [5, 7, 8]. The Delaunay depth contours have also be studied in [3]. The contours have applications in quality illumination too, for example, they can be applied to good illumination [1, 2] as we can see in the following. In this paper, we focus on the construction of the first α-depth contour that is also known as the α-embracing contour.

The good α-illumination [2] is a generalization of the 1-good illumination [1]. A point p in the plane is well α-illuminated by a set S of n point lights if there is, at least, one light interior to each wedge centered at p with a given angle α. According to the definition of α-depth, points well α-illuminated have α-depth greater or equal to 1. In the 1-good illumination, the convex hull separates the points that are 1-well illuminated from those which are not. The border of the convex hull is a contour that acts as a barrier between the 1-well illuminated points and the rest. A similar structure for the good α-illumination is called the α-embracing contour or the first α-depth contour and is defined next.

Definition 1.2. Let S be a set of n points in the plane and α a given angle ($0 \leq \alpha \leq \pi$). The α-embracing contour of S is the border of the region of points in the plane which are well α-illuminated.

We are going to prove that the α-embracing contour is composed by arcs of circumferences. The α-*arcs* defined in the following definition are the key to the α-embracing contour construction.

Definition 1.3. Let S be a set of n points in the plane, p and q two consecutive vertices of the convex hull of S and $a, b \in S$. The locus of points in the plane that see the line segment $\overline{ab}$ with a given angle α is composed by two arcs of circumferences with extreme points a and b. If $\overline{ab}$ is not perpendicular to $\overline{pq}$, we call one of these two arcs the α-arc of $\overline{ab}$ with respect to $\overline{pq}$: the one which is above the line containing a and b, if we consider a coordinate system in which p and q belong to the x-axis and the points of S have non negative y-coordinate. Such arc will be denoted by $\widehat{ab}$.

There is an example of Definition 1.3 in Figure 1(c). In this paper we will prove that the α-embracing contour consists on a linear number of pieces of α-arcs. There is an example of this structure in Figure 2(b) and it also proves the next property.

(a) (b)

Figure 2: (a) Only the points in the two grey areas are well $\frac{\pi}{2}$-illuminated. (b) The $\frac{\pi}{2}$-embracing contour of the set S which is not connected.

Proposition 1.4. *The α-embracing contour of a set S is not necessarily connected.*

In the next section we study the α-embracing contour. We prove that it has linear size and show how to compute it in quadratic time and linear space.

2 Construction of the α-embracing contour

This section is devoted to the construction of the first α-depth contour that will be called α-embracing contour from now on. Let S be a set of n points in the plane and α a given angle. The construction of the α-embracing contour is based on the next lemmas.

Lemma 2.1. *If two α-arcs associated with the same convex hull edge have a common extreme point $s_i \in S$ and intersect at $v \neq s_i$ then v is collinear with the other two extremes of the α-arcs.*

Proof. Every point on an α-arc is the apex of an α-wedge such that each ray goes through an extreme point of the corresponding arc. If two α-arcs associated with the same convex hull edge have a common extreme point $s_i \in S$ and intersect at $v \neq s_i$ (see Figure 3 (a)), then the wedge with apex at v and one ray through s_i and the other through any of the other extreme points of the two α-arcs is an α-wedge. Then v and the extreme points of the α-arcs different from s_i are collinear. □

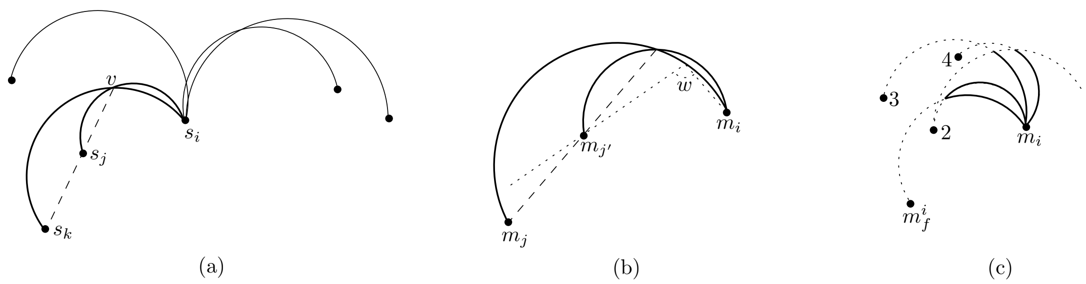

Figure 3: (a) The α-arcs $\widehat{s_i s_j}$ and $\widehat{s_i s_k}$ intersect at v and v, s_j and s_k are collinear. (b) The wedge w has an angle bigger than α only when m_j is an interior point to w. (c) The pieces of the α-arcs from m_i to an extreme of $M^l_{m_i}$ up to the first intersection with another arc are ordered according to those extremes.

Let $M_{s_k s_{k+1}} = \{m_1, \ldots, m_s\}$ be a set of unoriented α-maxima points of S (points of S that allow an empty α-wedge centered at them and can be computed by an algorithm by Avis et. al [4]) enclosed in the region delimited by the line segment $\overline{s_k s_{k+1}}$ and the α-arc $\widehat{s_k s_{k+1}}$, where $m_1 = s_k$ and $m_s = s_{k+1}$. For every point m_i consider the perpendicular line to $\overline{s_k s_{k+1}}$ through m_i and the partition of the set $M_{s_k s_{k+1}} \setminus \{m_i\}$ in $M^l_{m_i}$ and $M^r_{m_i}$. The set $M^l_{m_i}$ ($M^r_{m_i}$) contains the points on the left (right) of the perpendicular line. Let $m^i_f \in M^l_{m_i}$ be the first point around m_i in clockwise order and $m^i_l \in M^r_{m_i}$ the last.

Lemma 2.2. *Let $m_i \in M_{s_k s_{k+1}}$ be a vertex of the α-embracing contour. From this vertex, the following pieces of the α-arcs that are part of the α-embracing contour are:*

1. *the piece of the α-arc $\widehat{m_i m^i_f}$ from m_i ($i \neq 1$) up to the first intersection with another α-arc that has one of its extreme points in $M^l_{m_i}$*
2. *the piece of the α-arc $\widehat{m_i m^i_l}$ from m_i ($i \neq s$) up to the first intersection with another α-arc that has one of its extreme points in $M^r_{m_i}$.*

Proof. Let m_j and $m_{j'}$ be two points of $M^l_{m_i}$, m_j is first in clockwise order around m_i (similar arguments hold for two points of $M^r_{m_i}$).

Any wedge w with apex at a point of $\widehat{m_i m_j}$ and with one ray through m_i and the other through $m_{j'}$ has an angle bigger than α if and only if that wedge contains m_j in its interior. This situation happens when the wedge apex and m_i are in the same half-plane defined by the support line through $\overline{m_j m_{j'}}$ (see Figure 3(b)). That is, since m_j, $m_{j'}$ and $\widehat{m_i m_j} \cap \widehat{m_i m_{j'}}$ are collinear by Lemma 2.1, if the apex of a wedge lies at a point of the piece of the arc $\widehat{m_i m_j}$ from m_i up to $\widehat{m_i m_j} \cap \widehat{m_i m_{j'}}$ and it has one ray through m_i and the other through $m_{j'}$, its angle is bigger than α. Therefore the apex of an α-wedge with a ray through m_i and the other through $m_{j'}$ must be out of the region enclosed by $\widehat{m_i m_{j'}}$ and $\widehat{m_i m_j}$.

If there is not an intersection $\widehat{m_i m_j}$ and $\widehat{m_i m_{j'}}$, the arc $\widehat{m_i m_j}$ appears first in the clockwise order around m_i. The same happens when the two arcs intersect, $\widehat{m_i m_j}$ is still the first to appear from m_i up to their intersection point. If another α-arc not having m_i as an extreme point cuts $\widehat{m_i m_j}$ and $\widehat{m_i m_{j'}}$ on points of the pieces of the α-arcs from m_i up to $\widehat{m_i m_j} \cap \widehat{m_i m_{j'}}$, then the pieces from m_i up to this new cut have the same order than the previous pieces.

The pieces of the α-arcs from m_i to an extreme of the same subset ($M^l_{m_i}$ or $M^r_{m_i}$) up to the first intersection with some arc are ordered according to those extremes (see Figure 3(c)). Then if m_i

is a vertex of the α-embracing contour, only the pieces of the arcs $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ are part of the embracing contour. The pieces of these two arcs from m_i are only part of the embracing contour until they intersect with another arc with an extreme in the same set ($M^l_{m_i}$ for $\widehat{m_i m_f^i}$ and $M^r_{m_i}$ for $\widehat{m_i m_l^i}$). This is only true up to the first intersection because the order of the arcs swaps there. $\square$

Let $L_{s_k s_{k+1}}$ be the set of lines connecting m_i and m_f^i and m_i and m_l^i, for every point $m_i \in M_{s_k s_{k+1}}$.

Lemma 2.3. *A vertex of the α-embracing contour is either a point of a line of $L_{s_k s_{k+1}}$ or an intersection point between $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ for some $m_i \in M_{s_k s_{k+1}}$. Furthermore, every line of $L_{s_k s_{k+1}}$ contains one vertex of the first contour that does not belong to $M_{s_k s_{k+1}}$.*

Proof. First we prove that all the vertices of the α-embracing contour lie on a line of $L_{s_k s_{k+1}}$ or they are an intersection between $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ for some $m_i \in M_{s_k s_{k+1}}$.

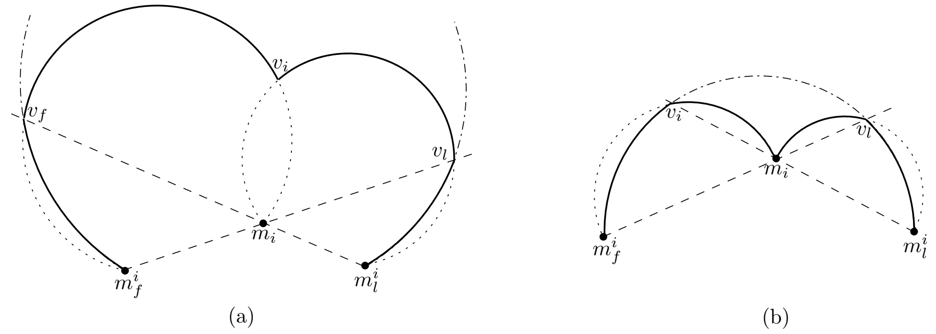

Figure 4: In both figures the α-embracing contour is represented in a solid trace. (a) The point m_i is not a vertex of the α-embracing contour but the intersection of the arcs $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ is. There are two other vertices that lie on two lines of $L_{s_k s_{k+1}}$ and are the result of the intersection of the two previous arcs with the arc $\widehat{m_f^i m_l^i}$. (b) The point m_i is a vertex of the α-embracing contour and there are two other vertices that lie on two lines of $L_{s_k s_{k+1}}$ and are the result of the intersection of the two previous arcs with the arc $\widehat{m_f^i m_l^i}$.

By the definition of the set $L_{s_k s_{k+1}}$, all the points of $M_{s_k s_{k+1}}$ lie on some line of the set $L_{s_k s_{k+1}}$ and they are possible vertices of the first contour since they are unoriented α-maxima [4]. If the α-arcs $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ intersect each other, that means that there is an empty α-wedge whose apex is at one of the arcs from m_i up to some intersection point, v_f of v_l, with some other arc (v_f for the arc with the extreme point m_f^i and v_l for the other). The α-wedge apex can only be located at any piece of the arc $\widehat{m_i m_f^i}$ ($\widehat{m_i m_l^i}$) from m_i to v_f (v_l). The α-wedge is empty when its apex is located at the intersection of both arcs, v_i. If the apex moves upwards along $\overrightarrow{m_i v_i}$ then one of the extremes of the arcs will become an interior point of the α-wedge. So the point where the apex is located no longer is a point of the α-embracing contour. If the apex moves on the opposite direction, then the empty wedge has an angle bigger than α, so v_i is a vertex of the embracing contour. By Lemma 2.2, the arcs $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ are also part of the embracing contour (see Figure 4).

Suppose that there are other vertices of the α-embracing contour different from the points of $M_{s_k s_{k+1}}$ and that are not intersections between $\widehat{m_i m_f^i}$ and $\widehat{m_i m_l^i}$ for some $m_i \in M_{s_k s_{k+1}}$. We will see that these vertices are points of some line of $L_{s_k s_{k+1}}$ and are located at the intersection of the α-arcs with extreme points of the set $M_{s_k s_{k+1}}$. Moreover these arcs have a common extreme. On the contrary, suppose that the vertex of the α-embracing contour is an intersection point between two α-arcs without a common extreme. Since these arcs cross, then the α-wedge with apex in this intersection point and a ray through one extreme of one of the arcs contains a point. This interior point is an extreme of the other arc. In this case, the α-wedge is not empty which contradicts the fact that it is a vertex of the α-embracing contour.

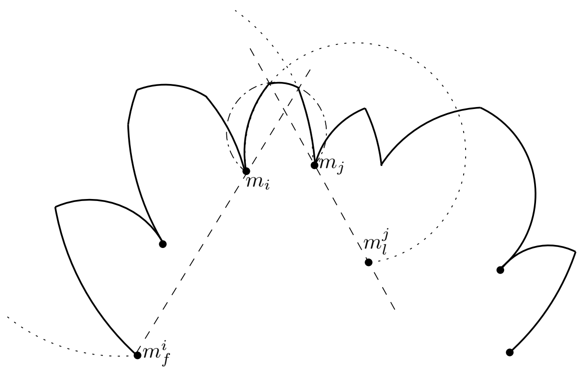

Figure 5: The line connecting m_i and m_f^i cuts the α-arc $\widehat{m_i m_j}$ in a vertex of the α-embracing contour. The line connecting m_j and m_l^j cuts α-arc $\widehat{m_i m_j}$ in another vertex.

Hence we can suppose that a vertex of the α-embracing contour is an intersection point between two α-arcs with a common extreme. Applying Lemma 2.1, the intersection between these arcs is collinear with the other two extreme points of the arcs. Now we show that one of these two points is the first or the last in clockwise order around the other. In Figure 5, we can see an example where m_l^j is the last point in clockwise order around m_j and m_f^i is the first in clockwise order around m_i. Otherwise, consider the α-wedge apex at the vertex of the embracing contour (the intersection point of the arcs). One of its rays goes through the common extreme of the arcs and the other goes through the two extremes that are collinear with the vertex. If one of the extremes were not the first or the last around the other, then it would have to be interior to the α-wedge. This contradicts the fact that the α-wedge apex is a vertex of the embracing contour. So the vertex lies on a line connecting two points of $M_{s_k s_{k+1}}$, where one of them is the last or the first in clockwise order around the other. That is, the vertex lies on a line of $L_{s_k s_{k+1}}$.

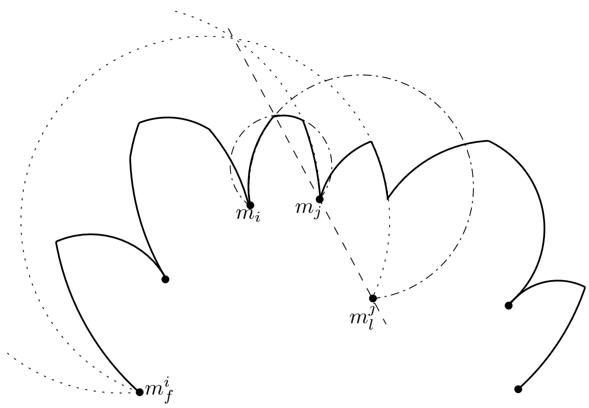

Figure 6: The line $\overline{m_j m_l^j}$ cuts the intersection between $\widehat{m_i m_j}$ and $\widehat{m_i m_l^j}$ and it also cuts the intersection between $\widehat{m_j m_f^i}$ and $\widehat{m_l^j m_f^i}$. These α-arcs share an extreme point (m_i in the first case and m_f^i in the second) while the other extremes are m_j and m_l^j. The first of these two cuts is a vertex of the α-embracing contour.

So now we prove that every line of $L_{s_k s_{k+1}}$ contains a vertex of the α-embracing contour that does not belong to $M_{s_k s_{k+1}}$. Let a line of $L_{s_k s_{k+1}}$ be defined by m_j and m_l^j (analogously m_j and m_f^j). Let us compute the cuts between this line and the intersection points between two α-arcs with extremes in $M_{s_k s_{k+1}}$ (see Figure 6). These cuts are intersection points between pairs of α-arcs with a common extreme point, while the other extreme points are m_j and m_l^j.

Consider the closest of these cuts to $\overline{s_k s_{k+1}}$, as long as they are different from m_j and m_l^j. Next we prove that if we choose this cut to locate the α-wedge apex with one ray through the common extreme of the arcs, m_i, and the other through m_j (or m_l^j since they are collinear), then this α-wedge is empty. Otherwise suppose that m_k is an interior point of the wedge. Note that by definition, m_k cannot be after m_l^j in the clockwise order around m_j. Then both $\widehat{m_j m_i}$ and $\widehat{m_j m_k}$ have an extreme in $M_{m_j}^l$. By construction, the α-wedge apex is located at the closest cut to $\overline{s_k s_{k+1}}$. The apex is at the intersection between the line $\overline{m_j m_l^j}$ and the intersection between the two α-arcs. Then the intersection between the α-arcs $\widehat{m_k m_j}$ and $\widehat{m_k m_l^j}$ is located along the line $\overline{m_j m_l^j}$ by Lemma 2.1 (see Figure 7). However the intersection point of these two α-arcs is closer to $\overline{s_k s_{k+1}}$ than the wedge apex. This contradicts the fact that the wedge apex is the closest point to $\overline{s_k s_{k+1}}$. So the α-wedge is empty when located at the closest cut to $\overline{s_k s_{k+1}}$. If we move the wedge apex from the intersection point along $\overrightarrow{m_l^j m_j}$ then one extreme point of the arcs becomes an interior point to the wedge. If we move the apex on the opposite direction, then the empty wedge has an angle bigger than α. That is, the closest cut to $\overline{s_k s_{k+1}}$ is a vertex of the α-embracing contour. So every line of $L_{s_k s_{k+1}}$ contains a vertex of the α-embracing contour that does not belong to $M_{s_k s_{k+1}}$. □

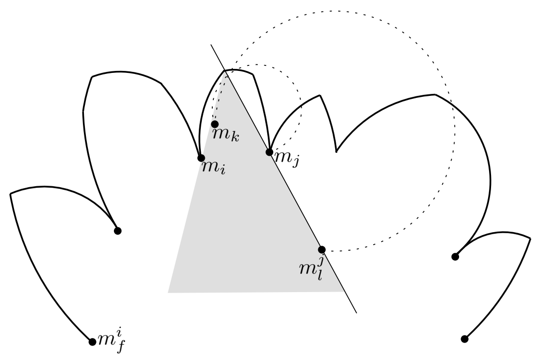

Figure 7: The intersection of the α-arcs $\widehat{m_k m_j}$ and $\widehat{m_k m_l^j}$ is located along the line $\overline{m_j m_l^j}$.

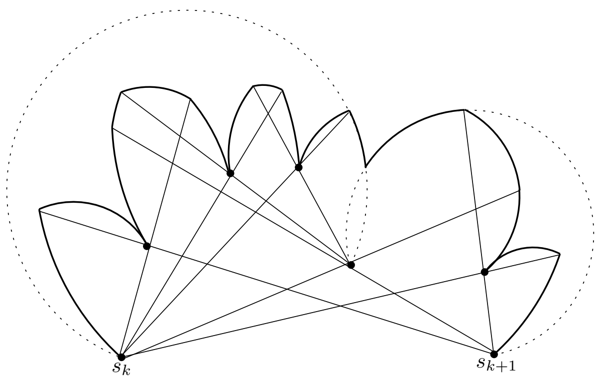

Figure 8: Every line of $L_{s_k s_{k+1}}$ contains a vertex of the α-embracing contour. In this example we can see the different kinds of vertices that appear in the α-embracing contour.

Proposition 2.4. *The α-embracing contour of a set S of n points has linear complexity.*

Proof. The α-embracing contour restricted to an edge $\overline{s_k s_{k+1}}$ of the CH(S) is a connected chain of α-arcs. As consequence of Lemma 2.3, we have seen that the vertices of the α-embracing contour restricted to $\overline{s_k s_{k+1}}$ can only be of three kinds: points of $M_{s_k s_{k+1}}$, intersection points between the first and the last α-arcs around some point of $M_{s_k s_{k+1}}$ or points lying on lines of $L_{s_k s_{k+1}}$ (see Figure 8). The cardinal of the set $L_{s_k s_{k+1}}$ is twice the cardinal of the set $M_{s_k s_{k+1}}$. The union of the sets $M_{s_k s_{k+1}}$ is the set of all the unoriented α-maxima points of S, which has $\mathcal{O}(n)$ points. So the α-embracing contour has linear complexity. □

Next we present a quadratic algorithm to compute the α-embracing contour based on the previous lemmas.

Algorithm 2.5. CONSTRUCTION OF THE α-EMBRACING CONTOUR OF S.

INPUT: Set of points S in non-degenerate position and an angle $\alpha, 0 < \alpha \leq \pi$.

OUTPUT: The α-embracing contour of S.

1. Compute the angular order of each point of S with respect to all of the others.
2. Compute CH(S). Let $\{s_1, \ldots, s_{n_1}\} \subseteq S$ be the set of the points on the border of CH(S).
3. Compute the unoriented α-maxima points using the algorithm by Avis et al. [4].
4. For every segment $\overline{s_k s_{k+1}}$, compute the associated α-arc $s_k \overset{\frown}{s}_{k+1}$.
5. Associate the set of unoriented α-maxima points that are in the enclosed region delimited by $\overline{s_k s_{k+1}}$ and $s_k \overset{\frown}{s}_{k+1}$ to every line segment $\overline{s_k s_{k+1}}$. Let $M_{s_k s_{k+1}}$ be that set of points.
6. For every point $m_i \in M_{s_k s_{k+1}}$, consider the first and last points of $M_{s_k s_{k+1}} \setminus \{m_i\}$ in clockwise order around m_i. Construct the set of lines $L_{s_k s_{k+1}}$, connecting each point m_i to the first and last points around it. (Note that the last point around s_k is s_{k+1} and the first around s_{k+1} is s_k).
7. For every point $m_i \in M_{s_k s_{k+1}}$, construct the first and the last α-arcs from m_i up to the first intersection with some line of the set $L_{s_k s_{k+1}}$.
8. If the chain of α-arcs just constructed is connected, then it is the α-embracing contour of S. If the chain of α-arcs is disconnected, suppose that the α-arc from m_i is disconnected. So this α-arc intersects a line of $L_{s_k s_{k+1}}$. Take the point $m_j \in M_{s_k s_{k+1}}$ that lies on that line and is not an extreme point of the previous α-arc. Construct the α-arc $\widehat{m_i m_j}$ up to the first intersection with some other line of the set $L_{s_k s_{k+1}}$ (if it exists). Repeat this process recursively until a connected chain of arcs is achieved.

As a consequence of the preceding paragraphs, we can state the following.

Theorem 2.6. *The α-embracing contour of a set S of n points in the plane can be constructed in $\mathcal{O}(n^2)$ time in the worst case and $\mathcal{O}(n)$ space.*

3 Conclusions and Future Work

We introduced the notion of α-depth and the first α-depth contour, also known as the α-embracing contour, as the contour that separates points in the plane that are well α-illuminated from the rest. We proved that the α-embracing contour consists on a linear number of pieces of α-arcs and presented an algorithm to compute the α-embracing contour of a point set in the plane that runs in $\mathcal{O}(n^2)$ time and $\mathcal{O}(n)$ space.

The study of all the α-depth contours and their properties is ongoing work, as well as the comparison between the α-depth contours and other depth contours.

4 Acknowledgments

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